Long-range dependence

Kechagias Stefanos

University of North Carolina at Chapel Hill

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Outline

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PART A
INTRODUCTION TO TIME SERIES

Introduction to time series

- Time series is a set of observations x_t, each one being recorded at a specified time t.
- Discrete time series: The set T_0 of times at which observations are made is discrete.

Example (Population in the U.S.)

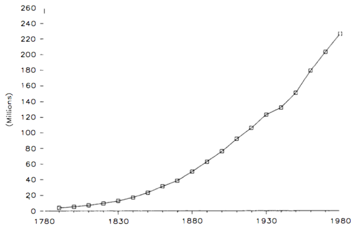


Figure 1.2. Population of the U.S.A. at ten-year intervals, 1790–1980 (U.S. Bureau of the Census).

t	x_t
1790	3,929,214
1800	5,308,483
1810	7,239,881
1820	9,638,453
$\vdots$	$\vdots$
1980	226,545,805

Mathematical model

Question: What mathematical model should we use in time series analysis?

- We can think of each observation x_t as a realized value of a certain random variable X_t.
- Then the time series $\{x_t, t \in T_0\}$ is the realization of the family of random variables $\{X_t, t \in T_0\}$.

Definition (Stochastic Process)

A *stochastic process* is a family of random variables $\{X_t, t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$.

- We will consider $T = \mathbb{Z}$.

Weak Stationarity

Definition (Stationarity)

A stochastic process $\{X_t\}_{t\in\mathbb{Z}}$ is (*weakly or second-order*) *stationary* if for any $t, s \in \mathbb{Z}$,

- $\mathbb{E}|X(t)|^2 < \infty$
- $\mathbb{E}X(t) = \mathbb{E}X(0)$
- $\mathrm{Cov}(X(t), X(s)) = \mathrm{Cov}(X(t-s), X(0))$

The time difference $t - s$ is called *time-lag*.

Time domain perspective

Consider a (weakly) stationary time series $\{X_t\}_{t\in\mathbb{Z}}$. In the time domain, one focuses on the functions

$$\begin{aligned} \gamma_X(h) &= \mathrm{Cov}(X_h, X_0) = \mathrm{Cov}(X_{t+h}, X_t), \quad \text{for all} \quad t, h \in \mathbb{Z} \\ \rho_X(h) &= \frac{\gamma_X(h)}{\gamma_X(0)}, \end{aligned}$$

ACV, ACVF

called *autocovariance function* (ACVF) and *autocorrelation function* (ACF). ACVF and ACF are measures of dependence in time series.

Sample counterparts of ACVF and ACF are the functions

$$\widehat{\gamma}_X(h) = \frac{1}{N}\sum_{t=1}^{N-|h|}(X_{t+|h|}-\overline{X})(X_{t+|h|}-\overline{X}), \quad \widehat{\rho}_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}, \quad |h| \leq N-1.$$

Basic Properties

1. (Symmetry) $\rho_X(h) = \rho_X(-h)$, $h \in \mathbb{Z}$
2. (Range) $|\rho_X(h)| \leq 1$, $h \in \mathbb{Z}$
3. (Interpretation) $\rho_X(h)$ close to -1, 0 and 1 correspond to strong negative, weak and strong positive correlation, respectively, in time series at lag h.

Time series examples

Example 1. (*White Noise*.) A time series $X_n = Z_n, n \in \mathbb{Z}$ is called *White Noise*, denoted $WN(0, \sigma^2)$, if $\mathbb{E}Z_n = 0$ and

$$\gamma_Z(h) = \begin{cases} \sigma^2, & \text{if } h = 0, \\ 0, & \text{if } h \neq 0, \end{cases} \qquad \rho_Z(h) = \begin{cases} 1, & \text{if } h = 0, \\ 0, & \text{if } h \neq 0. \end{cases}$$

Example 2. (*MA(1)*). A time series is called a *Moving Average of order 1* (MA(1)) if it is given by

$$X_n = Z_n + \theta Z_{n-1}, \quad n \in \mathbb{Z},$$

where $Z_n \sim WN(0, \sigma^2)$. Observe that,

$$\gamma_X(h) = \mathbb{E}X_h X_0 = \mathbb{E}(Z_h + \theta Z_{h-1})(Z_0 + \theta Z_{-1}) = \begin{cases} \sigma^2(1+\theta^2), & h = 0, \\ \sigma^2\theta, & h = 1, \\ 0, & h \geq 2, \end{cases}$$

and hence

$$\rho_X(h) = \begin{cases} 1, & h = 0, \\ \frac{\theta}{1+\theta^2}, & h = 1, \\ 0, & h \geq 2. \end{cases}$$

Example 3. (*AR(1)*). A (weakly) stationary time series $\{X_n\}_{n\in\mathbb{Z}}$ is called *Autoregressive of order 1* (AR(1)) if it satisfies the AR(1) equation

$$X_n = \phi X_{n-1} + Z_n, \quad n \in \mathbb{Z},$$

where $Z_n \sim WN(0, \sigma^2)$. To see that AR(1) time series exists, suppose $|\phi| < 1$ and notice that,

$$\begin{aligned} X_n &= \phi^2 X_{n-2} + \phi Z_{n-1} + Z_n \\ &= \phi^m X_{n-m} + \phi^{m-1} Z_{n-(m-1)} + \ldots + Z_n = \sum_{m=0}^{\infty} \phi^m Z_{n-m}. \end{aligned}$$

The time series is well defined in the $L^2(\Omega)$-sense because,

$$\mathbb{E}\left(\sum_{m=n_1}^{n_2}\phi^m Z_{n-m}\right)^2 = \sum_{m=n_1}^{n_2}\phi^{2m}\sigma^2 \to 0, \quad \text{as} \quad n_1, n_2 \to \infty,$$

for $|\phi| < 1$. Also we can easily calculate,

$$\gamma_X(h) = \sigma^2 \frac{\phi^{|h|}}{1-\phi^2}, \quad \rho_X(h) = \phi^{|h|}.$$

Example 4. (*ARMA(p,q)*) A (weakly) stationary time series $\{X_n\}_{n\in\mathbb{Z}}$ is called *Autoregressive moving average of orders p and q* (ARMA(p,q)) if it satisfies the equation

$$X_n - \phi_1 X_{n-1} - \ldots - \phi_p X_{n-p} = Z_n + \theta_1 Z_{n-1} + \ldots + \theta_q Z_{n-q},$$

where $Z_n \sim WN(0, \sigma^2)$. It is convenient to express the conditions in terms of the so-called backshift operator B, defined as

$$B^k X_n = X_{n-k}, \quad B^0 = I, n \in \mathbb{Z}.$$

Spectral domain

With this notation, the ARMA(p, q) equation becomes

$$\phi(B)X_n = \theta(B)Z_n,$$

where

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p, \quad \theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$$

are the so-called characteristic polynomials.

Spectral domain perspective

In the spectral domain the focus is on the function

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_X(h), \quad \lambda \in (-\pi, \pi],$$

called spectral density. Observe that f_X is well defined pointwise when $\gamma_X \in L^1(\mathbb{Z})$.

Spectral domain

Example 1. (White Noise cont'd). If $Z_n \sim WN(0, \sigma^2)$ then

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}, \ \lambda \in (-\pi, \pi].$$

Example 2. (AR(1) cont'd). If $\{X_n\}_{n\in\mathbb{Z}}$ is AR(1) time series with $|\phi| < 1$ and $\gamma_X(h) = \sigma^2\phi^h/(1-\phi^2)$, then

$$\begin{aligned} f_X(\lambda) &= \frac{\sigma^2}{2\pi(1-\phi^2)}\left(1 + \sum_{h=1}^{\infty}(e^{-ih\lambda} + e^{ih\lambda})\phi^h\right) \\ &= \frac{\sigma^2}{2\pi(1-\phi^2)}\left(1 + \frac{\phi e^{-i\lambda}}{1-\phi e^{-i\lambda}} + \frac{\phi e^{i\lambda}}{1-\phi e^{i\lambda}}\right) \\ &= \frac{\sigma^2}{2\pi}\frac{1}{|1-\phi e^{-i\lambda}|^2}. \end{aligned}$$

Spectral domain

Basic properties

1. *Symmetry* $f_X(\lambda) = f_X(-\lambda)$
2. *Non-negative* $f_X(\lambda) \geq 0$. For
3. *Inverse discrete Fourier transform*

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda) d\lambda.$$

A sample counterpart to the spectral density is defined by

$$\frac{1}{2\pi} \sum_{|h|<N} \widehat{\gamma}_X(h) e^{-ih\lambda} = \frac{1}{2\pi} \left| \sum_{n=1}^{N} X_n e^{-in\lambda} \right|^2 =: \frac{I_{X,N}(\lambda)}{2\pi},$$

with the first relation holding only at the so-called *Fourier frequencies*

$$\lambda = \lambda_k = \frac{2\pi k}{N} \quad \text{with} \quad k = -\left[\frac{N-1}{2}\right], \ldots, \left[\frac{N}{2}\right].$$

Spectral domain

$I_{X,N}$ is known as the *periodogram* and has the following properties:

1. (*Computational speed*). $I_{X,N}(\lambda_k)$ can be computed by fast Fourier Transform (FFT) in $O(N \log N)$ steps, supposing N can be factored out in many factors.
2. (*Statistical properties*). $I_{X,N}(\lambda)$ is not a consistent estimator for $2\pi f(\lambda)$, but is asymptotically unbiased. the periodogram needs to be smoothed out to become consistent.

PART B
LONG-RANGE DEPENDENCE

Slowly varying functions

Definition 1: A function L is slowly varying at infinity if it is positive and, for any $a > 0$,

$$\lim_{u \to \infty} \frac{L(au)}{L(u)} = 1.$$

A function L is slowly varying at zero if the function $L(1/u)$ is slowly varying at infinity.

Example: $L(u) = \log(u)$ is slowly varying at infinity.

We are now ready to introduce time series with long-range dependence. these will involve a parameter

$$d \in (0, 1/2),$$

and a slowly varying function L.

Definitions of long-range dependence

$d \in (0, 1/2), L_1, L_2$ slowly varying at infinity, L_3 slowly varying at zero.

Condition I. $X_n = \mu + \sum_{k=0}^{\infty} \psi_k \epsilon_{n-k}$ with $\{\epsilon_n\}_{n\in\mathbb{Z}} \sim WN$ and

$$\psi_k = L_1(k)k^{d-1}, \quad \text{as } k \to \infty.$$

Condition II.

$$\gamma(k) = L_2(k)k^{2d-1}, \quad \text{as } k \to \infty.$$

Condition III.

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| = \infty.$$

Condition IV.

$$f(\lambda) = L_3(\lambda)\lambda^{-2d}, \quad \text{as } \lambda \to 0.$$

In statistical inference, all slowly varying function are replaced by constants (in the asymptotic sense, e.g. $L_2(u) \sim c_2$).

Definitions of long-range depdendence

Definition 2: A second-order stationary time series $X = \{X_n\}_{n\in\mathbb{Z}}$ is called *long-range dependent* (LRD, in short) if one of the non-equivalent conditions IIV above holds. The parameter$d \in (0, 1/2)$ is called a *long-range dependence (LRD) parameter.*

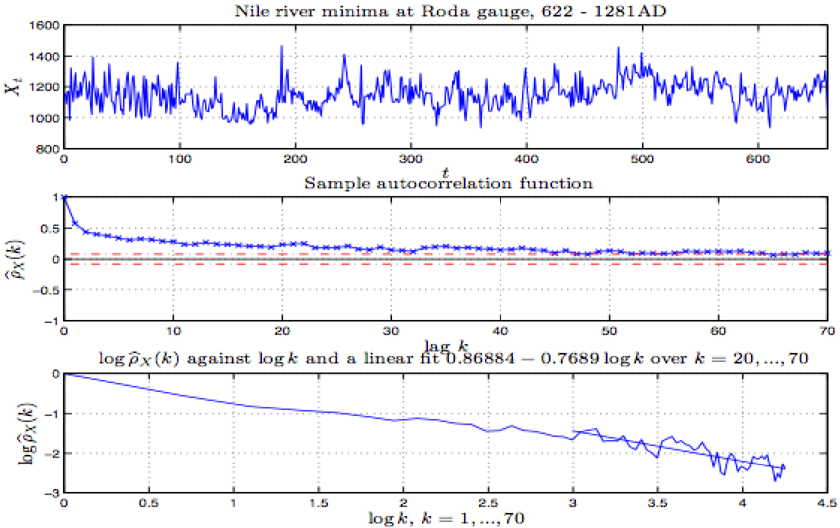
Other names: Long memory, strongly dependent, $1/f(1/\lambda)$ noise, colored noise, burstiness, ...

Definition 3: A second-order stationary time series $X = \{X_n\}_{n\in\mathbb{Z}}$ is called *short-range dependent* (SRD, in short) if

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty.$$

Other names: short memory, weakly dependent, ...

Some real examples of LRD series



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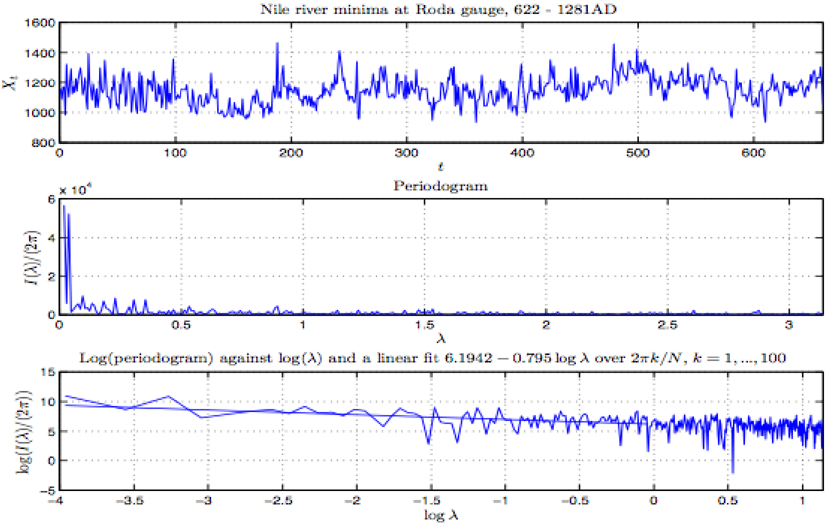

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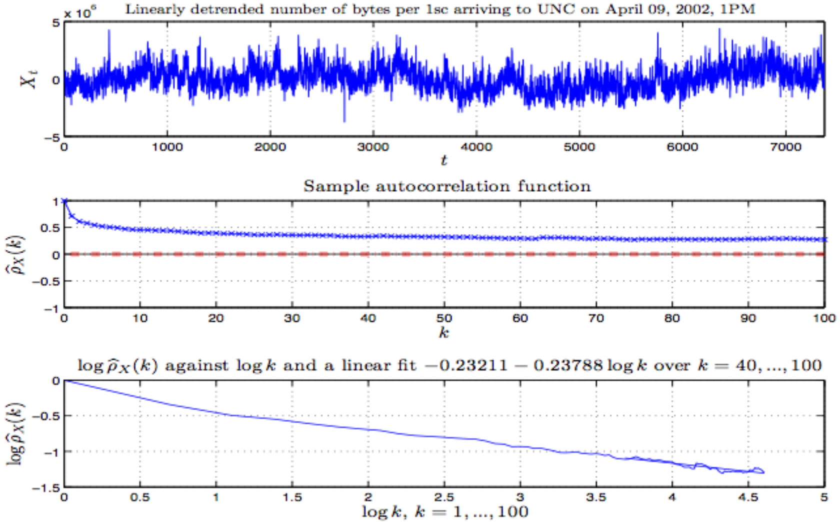
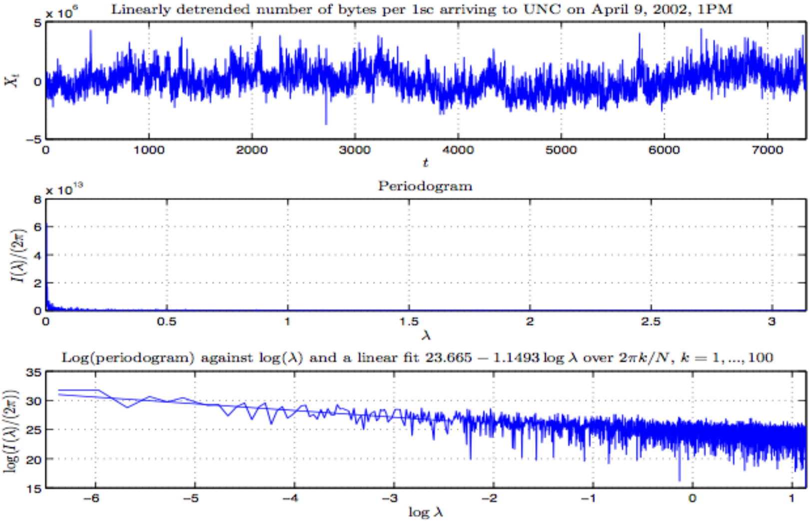


Some real examples of LRD series



Cond. IV $\Rightarrow$ III

As mentioned earlier Conditions I-IV are not equivalent in general. We will examine briefly how they are related.

Fact: If $\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty$, then the series $X = \{X_n\}_{n\in\mathbb{Z}}$ has a continuous spectral density $f(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-ik\lambda}\gamma(k)$.

If cond. III fails, that is, $\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty$, the fact above implies that $f(\lambda)$ is continuous and $f(0) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \gamma(k) < \infty$. But by cond. IV, $\lim_{\lambda\to 0} f(\lambda) = \lim_{\lambda\to 0} L_3(\lambda)\lambda^{-2d} = \infty$. Hence a contradiction, showing that cond. IV implies III.

Definition: A slowly varying function L on $[0,\infty)$ is called *quasi-monotone* if it is of bounded variation on any compact interval of $[0,\infty)$ and if for all $\delta > 0$,

$$\int_x^{\infty} u^{-\delta}|dL(u)| = O(x^{-\delta}L(x)), \quad \text{as} \quad x \to \infty.$$

Comparing cond. II and IV

Cond. IV $\Rightarrow$ II: Informally,

$$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda = \int_{-\pi}^{\pi} e^{ik\lambda} L_3(\lambda)\lambda^{-2d} d\lambda$$

$$= k^{2d-1} L_3(\frac{1}{k}) \int_{-k\pi}^{k\pi} e^{iz}|z|^{-2d} \frac{L_3(\frac{|z|}{k})}{L_3(\frac{1}{k})} dz$$

$$\approx k^{2d-1} L_3(\frac{1}{k}) \int_{-\infty}^{\infty} e^{iz}|z|^{-2d} dz$$

$$= k^{2d-1} L_3(\frac{1}{k}) 2\cos(\frac{\pi(1-2d)}{2})\Gamma(1-2d).$$

and hence $L_2(u) \sim L_3(1/u) 2\cos(\frac{\pi(1-2d)}{2})\Gamma(1-2d)$. These calculations can be justified assuming that L_3 is quasi-monotone. Otherwise the result is not in general even in the case $L_3(k) \sim c_3$.

Comparing cond. II and IV

Cond. II $\Rightarrow$ IV: Informally,

$$f(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-ik\lambda}\gamma(k) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-ik\lambda} L_2(k)k^{2d-1}$$

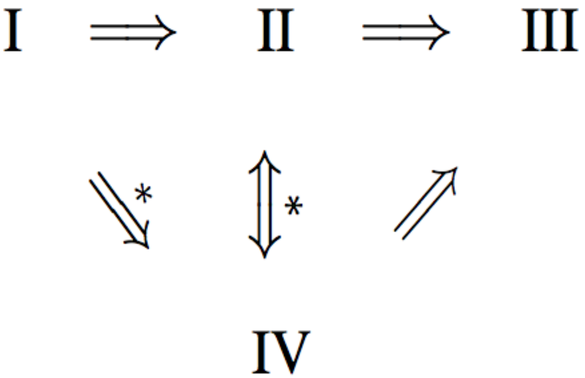
$$= \lambda^{-2d} L_2(\frac{1}{\lambda})\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-ik\lambda}\frac{L_2(k\lambda\frac{1}{\lambda})}{L_2(\frac{1}{\lambda})}(k\lambda)^{2d-1}\lambda$$

$$\approx \lambda^{-2d} L_2(\frac{1}{\lambda})\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iz} z^{2d-1} dz$$

$$= \lambda^{-2d} L_2(\frac{1}{\lambda})\frac{1}{2\pi}\Gamma(2d)\cos(\pi d),$$

thus $L_3(\lambda) \sim L_2(\frac{1}{\lambda})\frac{1}{\pi}\Gamma(2d)\cos(\pi d)$. These calculations can be justified assuming that L_2 is quasi-monotone. Otherwise the result is not in general even in the case $L_2(k) \sim c_2$.

Some real examples of LRD series



"$\overset{*}{\Longrightarrow}$" supposes that the slowly varying function is *quasi-monotone*

FARIMA$(0, d, 0)$

Example : If d is a non negative integer, then X_t is said to be an $ARIMA(0, d, 0)$ process if

$$(I - B)^d X_n = Z_n,$$

where $Z_n \sim WN(0, \sigma^2)$ and B is the backshift operator. In simple cases of d=1,2,.., this equation is:

$$d = 1 \ : \ (I - B)X_n = X_n - X_{n-1} = Z_n$$
$$d = 2 \ : \ (I - B)^2 X_n = (X_n - X_{n-1}) - (X_{n-1} - X_{n-2}) = Z_n$$

and so on when $d \geq 3$.

Remark: In many applications we want to difference the observed time series in order to achieve approximate stationarity. However even though differencing might seem appropriate, taking the first or second difference may be too strong!

Example cont'd: For other values of d we would like to interpret the solution to the ARIMA$(0, d, 0)$ as

$$X_n = (I - B)^{-d} Z_n = \sum_{j=0}^{\infty} b_j B^j Z_n = \sum_{j=0}^{\infty} b_j Z_{n-j},$$

where $b_j' s$ are the coefficients in the Taylor expansion of

$$(1 - z)^{-d} = 1 + dz + \frac{d(d+1)}{2!} z^2 + \frac{d(d+1)(d+2)}{3!} z^3 + \dots$$

$$= \sum_{j=0}^{\infty} \left(\prod_{k=1}^{j} \frac{k - 1 + d}{k} \right) z^j = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j)\Gamma(d)} z^d =: \sum_{j=0}^{\infty} b_j z^j.$$

The time series is well defined when $\sum_{j=0}^{\infty} b_j^2 < \infty$ which depends on the behavior of b_j as $j \to \infty$.

FARIMA$(0, d, 0)$

Example cont'd: By Stirling's formula $\Gamma(p) \sim \sqrt{2\pi} e^{-(p-1)}(p-1)^{p-1/2}$, as $p \to \infty$, we have

$$b_j = \frac{\Gamma(j+a)}{\Gamma(j+b)} \sim \frac{j^{d-1}}{\Gamma(d)}, \quad \text{as} \quad j \to \infty.$$

Then the time series is well defined if $2(d-1)+1 = 2d-1 < 0$ or $d < 1/2$.

Definition: The time series $X_n = (I-B)^{-d} Z_n = \sum_{j=0}^{\infty} b_j Z_{n-j}$ is called FARIMA$(0, d, 0)$ when $d < 1/2$.

Since $b_j \sim \frac{j^{d-1}}{\Gamma(d)}$ a FARIMA$(0, d, 0)$ series is LRD when $0 < d < 1/2$, in the sense of condition I and hence II and III.

FARIMA$(0, d, 0)$

Example cont'd:

Fact 1: FARIMA$(0, d, 0)$ series has spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left(2\sin\frac{\lambda}{2}\right)^{-2d} = \frac{\sigma^2}{2\pi}|1 - e^{-i\lambda}|^{-2d} \sim \frac{\sigma^2}{2\pi}|\lambda|^{-2d} \text{ as } \lambda \to 0.$$

Basic idea: A series $X_n = \sum_{j=0}^{\infty} b_j Z_{n-j}$ has spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left|\sum_{j=0}^{\infty} b_j e^{-ij\lambda}\right|^2$$

Here, $\sum_{j=0}^{\infty} b_j e^{-ij\lambda} = (1 - e^{-i\lambda})^{-d}$.

FARIMA$(0, d, 0)$

Fact 2: FARIMA$(0, d, 0)$ series has autocovariances

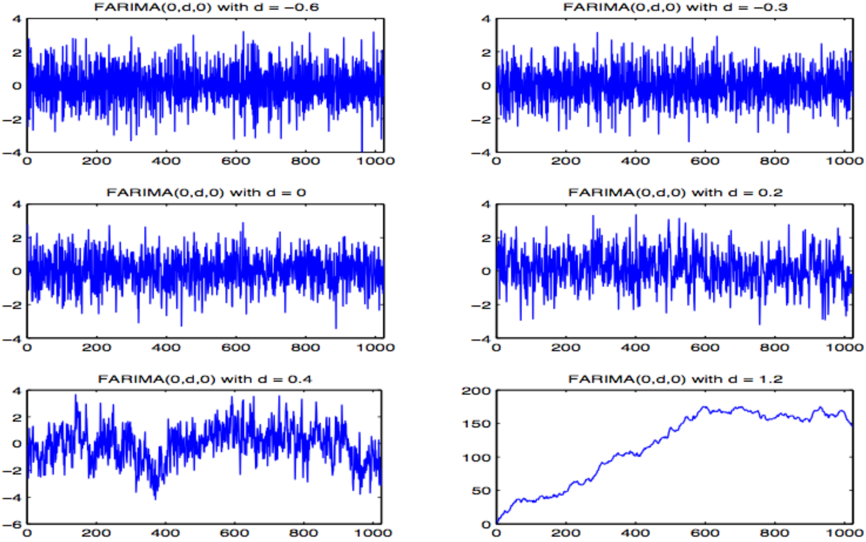
$$\gamma(k) = \sigma^2 \frac{(-1)^k \Gamma(1-2d)}{\Gamma(1-d)\Gamma(1-k-d)} \sim \sigma^2 \frac{\Gamma(1-2d) sin(\pi d)}{\pi} k^{2d-1}, \text{ as } k \to \infty.$$

Basic idea:

$$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda = \frac{\sigma^2}{2\pi} \int_0^{2\pi} \cos(k\lambda)(2\sin(\lambda/2))^{-2d} d\lambda$$

and the formula for the last integral is known.

FARIMA(0, d, 0)



FARIMA(0,d,0) with d = −0.6
FARIMA(0,d,0) with d = −0.3
FARIMA(0,d,0) with d = 0
FARIMA(0,d,0) with d = 0.2
FARIMA(0,d,0) with d = 0.4
FARIMA(0,d,0) with d = 1.2

Long memory stochastic volatility model (LMSV).

Consider the latent variable model for the return series r_t

$$r_t = \sigma_t v_t,$$

where v_t is an independent identically distributed series with mean zero and finite variance and σ_t^2 is given by,

$$\sigma_t = exp(h_t/2),$$

where h_t is a Gaussian long memory series independent of v_t.

Using the moment generating function of the Gaussian distribution it can be shown for the LMSV model that

$$\rho_{r_t^2}(j) \sim Cj^{2d-1}, \quad \text{as } j \to \infty.$$

PART C
MULTIVARIATE LRD

Multivariate LRD

We focus here on vector-valued ($\mathbb{R}^p$-valued), second order stationary times series $X = \{X_n\}_{n\in\mathbb{Z}}$.

- Autocovariance matrix function:

$$\gamma(h) = (\gamma_{jk}(h))_{j,k=1,\dots,p} = \mathbb{E}X_0X_h' - \mathbb{E}X_0\mathbb{E}X_h', \quad h \in \mathbb{Z}$$

- Spectral density matrix function (if it exists)

$$f(\lambda) = (f_{jk}(\lambda))_{j,k=1,..,p}, \quad \lambda \in (-\pi, \pi].$$

It satisfies

$$\int_{-\pi}^{\pi} e^{in\lambda} f(\lambda)d\lambda = \gamma(n), \quad n \in \mathbb{Z}.$$

Remark 1: The cross covariances $\gamma_{jk}(n)$ may not be equal to $\gamma_{jk}(-n)$ when $j \neq k$. Therefore in contrast to the univariate case, it is not true in general that

$$\gamma(n) = \gamma(-n), \quad n \in \mathbb{Z}. \tag{1}$$

If (1) holds, the time series X is called *time reversible*.

Remark 2: Since (1) may not hold, $f(\lambda)$ is in general, complex valued. $f(\lambda)$ is Hermitian symmetric, non negative definite and satisfies $f(-\lambda) = \overline{f(\lambda)}$, $\lambda \in [-\pi, \pi)$.

We are now ready to extend Conditions II and IV to the multivariate case. Let

$$D = \mathrm{diag}(d_1, \ldots, d_p) \quad \text{with} \quad d_j \in (0, 1/2), j = 1, \ldots, p.$$

Multivariate LRD

Condition II-v. The autocovariance matrix function of the time series $X = \{X_n\}_{n\in\mathbb{Z}}$ satisfies:

$$\gamma(n) = n^{D-(1/2)I} L_2(n) n^{D-(1/2)I}, \tag{2}$$

where L_2 is an $\mathbb{R}^{p\times p}$-valued function satisfying $L_2(u) \sim R$, as $u \to +\infty$, for some $p \times p$ matrix R. Equivalently we can write,

$$\gamma_{jk}(n) = L_{2,jk}(n) n^{(d_j+d_k)-1} \sim R_{jk} n^{(d_j+d_k)-1}, \quad \text{as} \quad n \to \infty.$$

Condition IV-v. The spectral density matrix function satisfies

$$f(\lambda) = \lambda^{-D} L_3(\lambda) \lambda^{-D^*}, \tag{3}$$

where L_3 is a $\mathbb{C}^{p\times p}$-valued, Hermitian symmetric, non-negative definite matrix function satisfying $L_3(\lambda) \sim G$, as $\lambda \to 0$, for some $p \times p$, Hermitian symmetric, non-negative definite matrix G. Equivalently,

$$f_{jk}(\lambda) = L_{3,jk}(\lambda) \lambda^{-(d_j+d_k)} \sim G_{jk} \lambda^{-(d_j+d_k)} =: g_{jk} e^{i\phi_{jk}} \lambda^{-(d_j+d_k)} \quad \text{as} \quad \lambda \to 0,$$

where $g_{jk} \in \mathbb{R}$ and $\phi_{jk} \in [-\pi, \pi)$.

Remarks:

1. The individual component series $\{X_n^j\}_{n\in\mathbb{Z}}$, $j = 1, \dots, p$, of a multivariate LRD series are LRD with parameters d_j, $j = 1, \dots, p$.
2. Note from (3) that $f(\lambda)$ is Hermitian, non negative definite. The entries ϕ_{jk} are referred to as *phase parameters*.
3. We supposed for simplicity that all slowly varying functions behave as constants.
4. The squared coherence function $\mathcal{H}_{jk}^2(\lambda) = |f_{jk}(\lambda)|^2/(f_{jj}(\lambda)f_{kk}(\lambda))$ satisfies $0 \le \mathcal{H}_{jk}^2(\lambda) \le 1$. As $\lambda \to 0$, this translates into

$$0 \le \lim_{\lambda\to 0} \frac{|G_{jk}|^2 \lambda^{-2(d_j+d_k)}}{G_{jj}\lambda^{-2d_j} G_{kk}\lambda^{-2d_k}} = \frac{|G_{jk}|^2}{G_{jj}G_{kk}} \le 1 \qquad (4)$$

and also explains why the choice of $\lambda^{-(d_j+d_k)}$ is natural for the cross-spectral density $f_{jk}(\lambda)$.

Comparing cond. II-v and IV-v

- Suppose that $L_{2,jk}$ are quasi-monotone. Then cond II-v implies cond IV-v with $G_{jk} = g_{jk} e^{i\phi_{jk}}$ given by

$$\phi_{jk} = -\arctan\left\{\frac{R_{jk} - R_{kj}}{R_{jk} + R_{kj}} \tan(\frac{\pi}{2}(d_j + d_k))\right\},$$

$$g_{jk} = \frac{\Gamma(d_j + d_k)(R_{jk} + R_{kj}) \cos(\frac{\pi}{2}(d_j + d_k))}{2\pi \cos(\phi_{jk})}.$$

- Suppose that $\Re L_{3,jk}$, $\Im L_{3,jk}$ are quasi-monotone. Then cond IV-v implies cond II-v with

$$R_{jk} = 2\Gamma(1-(d_j+d_k))\left\{\Re G_{jk} \sin(\frac{\pi}{2}(d_j+d_k)) - \Im G_{jk} \cos(\frac{\pi}{2}(d_j+d_k))\right\}$$

Linear Representation

Suppose that the series X_n has a linear representation of the form

$$X_n = \sum_{m=-\infty}^{\infty} \Psi_m Z_{n-m},$$

where $\{\Psi_m = (\psi_{m,jk})_{j,k=1,\ldots,p}\}_{m\in\mathbb{Z}}$, a sequence of real matrices such that

$$\psi_{m,jk} = L_{jk}(m)|m|^{d_j-1},$$

where $L(m) = \left(L_{jk}(m)\right)_{j,k=1,\ldots,p}$ is an $\mathbb{R}^{p\times p}$-valued function satisfying

$$L(m) \sim A^+ \quad \text{as} \quad m \to \infty, \quad \text{and} \quad L(m) \sim A^- \quad \text{as} \quad m \to -\infty,$$

for some $p \times p$ real matrices $A^+ = (a_{jk}^+)_{j,k=1,\ldots,p}$, $A^- = (a_{jk}^-)_{j,k=1,\ldots,p}$.

Then X_n is LRD.

Example

Consider the series

$$X_n = \sum_{m=0}^{\infty} \Psi_m Z_{n-m},$$

where $\{Z_n\}_{n\in\mathbb{Z}}$, and Ψ_m as earlier. Then the phase parameters appearing in the (j,k) element of the spectral density matrix of X_n have the form $\phi_{jk} = -(d_j - d_k)\frac{\pi}{2}$.

Question: What behavior should I consider for Ψ_m to get a causal represenation?

Linear combinations of

$$c_m^{a,b} = m^{-b}\cos(2\pi m^a) \quad \text{and} \quad s_m^{a,b} = m^{-b}\sin(2\pi m^a) \quad (?)$$

FARIMA(0,D,0)

- $M_+ = (m_{ij}^+)$, $M_- = (m_{ij}^-) \in \mathbb{R}^{p\times p}$.
- $\{Z_n\}_{n\in\mathbb{Z}}$ such that $\mathbb{E}Z_n = 0$ and $\mathbb{E}Z_n Z_m' = \sigma^2 I$ if $n = m$, and $= 0$ if $n \neq m$.

Define a multivariate FARIMA$(0, D, 0)$ series as

$$X_n = (I - B)^{-D} M_+ Z_n \; + \; (I - B^{-1})^{-D} M_- Z_n.$$

For the component wise spectral density I have

$$f_{jk}(\lambda) \sim c_{j,k} \lambda^{-(d_j + d_k)}, \text{ as } \lambda \to 0,$$

where

$$c_{j,k} = \frac{\sigma^2}{2\pi} \left(e^{-i(d_j - d_k)\frac{\pi}{2}} A_1 + e^{-i(d_j - d_k)\frac{\pi}{2}} A_2 + e^{i(d_j - d_k)\frac{\pi}{2}} A_3 + e^{i(d_j - d_k)\frac{\pi}{2}} A_4 \right)$$

and $A_1 = \sum_{t=1}^p m_{jt}^+ m_{kt}^+$, $A_2 = \sum_{t=1}^p m_{jt}^+ m_{kt}^-$, $A_3 = \sum_{t=1}^p m_{jt}^- m_{kt}^+$, $A_4 = \sum_{t=1}^p m_{jt}^- m_{kt}^-$.

Lots and Lots of Fun Stuff

- Self similar processes
- Ising model in two dimensions
- Infinite sourse Poisson model with Heavy tails

References

The results in part c are a joint work with my advisor Vladas Pipiras.

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Thank you!