# Definitions and representations of multivariate long-range dependent time series*†

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**Abstract**

The notion of multivariate long-range dependence is reexamined here from the perspectives of time and spectral domains. The role of the so-called phase parameters is clarified and stressed throughout. In particular, examples of causal (one-sided) representations of multivariate long-range dependent time series with general phase parameters are constructed. A multivariate extension of FARIMA series is introduced with explicit formulae for its autocovariance function.

## 1 Introduction

Long-range dependent (LRD, in short) time series models have been studied extensively in theory and have been popular in a wide range of applications (Beran (2013), Doukhan Oppenheim and Taqqu (2003), Giraitis, Koul and Surgailis (2012), Park and Willinger (2000), Robinson (2003)). They are defined as (second-order) stationary time series models satisfying one of the following non-equivalent conditions. In the time domain, the autocovariance function $\gamma(h) = \mathrm{Cov}(X_0, X_h)$ of a LRD time series $\{X_n\}_{n\in\mathbb{Z}}$ is such that

$$\gamma(k) = L_1(k)k^{2d-1}, \quad \text{as } k \to \infty, \tag{1.1}$$

where $d \in (0, 1/2)$ is the *long-range dependence (LRD) parameter* and $L_1$ is a slowly varying function at infinity. In the spectral domain, the spectral density function $f(\lambda)$ of $\{X_n\}_{n\in\mathbb{Z}}$ is such that

$$f(\lambda) = L_2(\lambda)\lambda^{-2d}, \quad \text{as } \lambda \downarrow 0, \tag{1.2}$$

where $L_2$ is a slowly varying function at 0. Another common way to define a LRD time series is through a causal (one-sided) linear representation

$$X_n = \mu + \sum_{k=0}^{\infty} \psi_k \epsilon_{n-k}, \tag{1.3}$$

where $\{\epsilon_n\}_{n\in\mathbb{Z}}$ is a white noise series, $\mu$ is a constant mean and the sequence $\{\psi_k\}_{k\geq 0}$ satisfies

$$\psi_k = L_3(k)k^{d-1}, \quad \text{as } k \to \infty, \tag{1.4}$$

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where $L_3$ is a slowly varying function at infinity.

In this work, we are interested in the notion of LRD for multivariate time series, that is, $\mathbb{R}^p$-valued time series $X_n = (X_n^1, \ldots, X_n^p)'$, where prime indicates transpose. Second-order stationary multivariate series are now characterized by matrix-valued autocovariance function $\gamma(h) = \mathbb{E}X_0X_h' - \mathbb{E}X_0\mathbb{E}X_h'$ and matrix-valued spectral density $f(\lambda)$. In the multivariate case, $\gamma(h)$ does *not* necessarily satisfy the symmetry (time-reversibility) condition $\gamma(h) = \gamma(-h)$ and, moreover, $f(\lambda)$ can have complex-valued entries in general (Hannan (1970), Reinsel (1997), Lütkepohl (2005)). Several forms of multivariate LRD, not surprisingly, have already been considered in the literature. The goal of this work is to clarify this notion. As will be seen below, there are a number of new interesting issues that arise in the multivariate but not in the univariate case, and which have not been studied in greater detail yet.

The most general form of the *bivariate* LRD appears in Robinson (2008) who supposed that the spectral density matrix satisfies

$$f(\lambda) \sim \begin{pmatrix} \omega_{11}|\lambda|^{-2d_1} & \omega_{12}|\lambda|^{-(d_1+d_2)}e^{-i\mathrm{sign}(\lambda)\phi} \\ \omega_{21}|\lambda|^{-(d_1+d_2)}e^{i\mathrm{sign}(\lambda)\phi} & \omega_{22}|\lambda|^{-2d_2} \end{pmatrix}, \quad \text{as } \lambda \to 0, \tag{1.5}$$

where $\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22} \in \mathbb{R}$, $d_1, d_2 \in (0, 1/2)$ and $\phi \in (-\pi, \pi]$. The parameter $\phi$ is called a *phase* parameter and is unique to the bivariate LRD case (see also Section 2 below). It controls asymmetry (time non-reversibility) in the series at large time lags. Many results of this work will be related directly to this parameter and its role. The definition of LRD with

$$\phi = 0 \tag{1.6}$$

was considered in Lobato (1999), Velasco (2003), Marinucci and Robinson (2003), Christensen and Nielsen (2006), Nielsen (2004, 2007). The value $\phi = 0$ is associated with LRD time series which are symmetric (time-reversible). The case of

$$\phi = \frac{\pi}{2}(d_1 - d_2) \tag{1.7}$$

is another special case and was considered in Lobato (1997), Robinson (2002, 2008), Shimotsu (2007), Nielsen (2011), Nielsen and Frederiksen (2011). For example, a natural extension of the FARIMA$(0, d, 0)$ series to the bivariate case as

$$\begin{pmatrix} (I-B)^{d_1} & 0 \\ 0 & (I-B)^{d_2} \end{pmatrix} X_n = \epsilon_n, \tag{1.8}$$

where $\{\epsilon_n\}$ is a bivariate white noise and $B$ is the backshift operator, corresponds to $\phi$ given by (1.7). The case of general $\phi$ is considered in Robinson (2008) as indicated above, and without referring to $\phi$ explicitely in Robinson (1995). Multivariate LRD models also appear in Robinson (1994), Chan and Terrin (1995), Marinucci and Robinson (2001, 2003), Robinson and Yajima (2002), Chen and Hurvich (2003, 2006) in the context of fractional cointegration, and in Achard, Bassett, Meyer-Lindenberg and Bullmore (2008), Wendt, Scherrer, Abry and Achard (2009) in the context of fractal connectivity.

We contribute to the understanding of the notion of multivariate LRD in the following three ways. First, we extend the definition (1.5) to the multivariate case and consider its analogue in the time domain (as in (1.1)) and relationships between them. This contribution is somewhat standard, but also necessary to set a proper foundation for dealing with multivariate LRD. Again, much of the discussion will focus on the role played by the phase parameters ($\phi$ in (1.5) in the bivariate case). In the bivariate case, similar results can be found in Robinson (2008).

Our second contribution is more original. It concerns linear representations of multivariate LRD series of the form

$$X_n = \sum_{k=-\infty}^{\infty} \Psi_k \epsilon_{n-k}, \tag{1.9}$$

where $\Psi_k$ are $p \times p$ matrices and $\{\epsilon_n\}$ is a $p$-variate white noise. Even more specifically, we are interested in *causal (one-sided)* representations, that is, (1.9) with $\Psi_k = 0$ when $k < 0$. It is not too difficult to construct *non-causal (two-sided)* representations (1.9) having general phase parameters in the definition of LRD by having $\Psi_k$ decay as suitable power-law functions as $k \to \infty$ and $k \to -\infty$. But it is not obvious how to construct such causal representations. For example, taking $\Psi_k$ to behave as a power-law function in a causal representation leads necessarily to the phase parameters $\frac{\pi}{2}(d_{j_1} - d_{j_2})$, where $d_{j_1}, d_{j_2}$ are the LRD parameters of component series (Section 3 below).

We show that *causal* multivariate LRD series of general phase can be constructed taking the elements of $\Psi_k$ as linear combinations of, what we will call, *trigonometric power-law coefficients*

$$\begin{aligned} c_k^{a,b} &= k^{-b}\cos(2\pi k^a), \\ s_k^{a,b} &= k^{-b}\sin(2\pi k^a), \qquad k \geq 0, \end{aligned} \tag{1.10}$$

where $0 < a < 1$ and $\frac{1}{2} < b \leq 1 - \frac{1}{2}a$. (By convention, $0^p = 0$ for $p \in \mathbb{R}$, so that $c_0^{a,b} = s_0^{a,b} = 0$.) The use of such coefficients can be traced back at least to Wainger (1965). What makes them special and relevant for LRD is that their discrete Fourier transform satisfies, for example,

$$\sum_{k=0}^{\infty} c_k^{a,b} e^{-ik\lambda} \sim c_1 |\lambda|^{-\frac{1-b-a/2}{1-a}} e^{i(c_2|\lambda|^{-\frac{a}{1-a}} - \frac{\pi}{4})}, \quad \text{as} \quad \lambda \downarrow 0, \tag{1.11}$$

where $c_1, c_2$ are two non-zero constants (Wainger (1965) and Appendix B below). Thus, even in the univariate case, the time series

$$X_n = \sum_{k=0}^{\infty} c_k^{a,b} \epsilon_{n-k} \tag{1.12}$$

is LRD (cf. (1.2)) with the LRD parameter

$$d = \frac{1 - b - a/2}{1 - a}. \tag{1.13}$$

Though the trigonometric power-law coefficients can be used to construct new univariate and multivariate causal LRD series, their statistical inference and practical usefulness remain to be explored in the future.

Lastly and third, we provide a natural multivariate extension of FARIMA$(0, d, 0)$ having a general phase. As indicated above, the extension (1.8) of FARIMA$(0, d, 0)$ series to the bivariate case has necessarily the phase parameter $\phi = (d_1 - d_2)\pi/2$. For example, a bivariate extension with a general phase can be obtained with

$$X_n = \begin{pmatrix} (I-B)^{-d_1} & 0 \\ 0 & (I-B)^{-d_2} \end{pmatrix} Q_+ \epsilon_n + \begin{pmatrix} (I-B^{-1})^{-d_1} & 0 \\ 0 & (I-B^{-1})^{-d_2} \end{pmatrix} Q_- \epsilon_n, \tag{1.14}$$

where $Q_+$, $Q_-$ are $2 \times 2$ matrices with real-valued entries. We provide explicit formulas for the autocovariance functions of this extension, including the multivariate case, in Section 5.

The structure of the paper is as follows. The definitions of multivariate LRD in the time and spectral domains are given in Section 1. Non-causal representations of multivariate LRD series are studied in Section 3. Section 4 concerns causal linear representations. Multivariate FARIMA series are considered in Section 5. Some technical proofs are moved to Appendix A, and the behavior of the Fourier series of the trigonometric power-law coefficients (1.10) is presented in Appendix B.

## 2 Definitions in the time and spectral domains

We begin with the definitions of multivariate LRD in the time and spectral domains, extending conditions (1.1) and (1.2). We shall suppose for simplicity that all slowly varying functions behave asymptotically as constants. This is the relevant case for statistical inference. Moreover, these slowly varying functions would appear below in a matrix form which, along a matrix regular variation, is only now receiving a closer look (e.g. Meerschaert and Scheffler (2013)).

The definitions below use the following notation. For $a > 0$ and a diagonal matrix $M = \mathrm{diag}(m_1, \ldots, m_p)$, we write $a^M = \mathrm{diag}(a^{m_1}, \ldots, a^{m_p})$. The autocovariance matrix function of a second-order stationary series $X = \{X_n\}_{n\in\mathbb{Z}}$ is defined as $\gamma(h) = (\gamma_{jk}(h))_{j,k=1,\ldots,p} = \mathbb{E}X_0X_h' - \mathbb{E}X_0\mathbb{E}X_h'$, $h \in \mathbb{Z}$, and the corresponding spectral density matrix function, if it exists, is denoted $f(\lambda) = (f_{jk}(\lambda))_{j,k=1,\ldots,p}$. For a matrix $A$, $A^*$ stands for its Hermitian transpose.

**Definition 2.1** (*Time domain*) A multivariate ($p$-vector) second-order stationary time series is LRD if its autocovariance matrix function satisfies:

$$\gamma(n) = n^{D-(1/2)I}R(n)n^{D-(1/2)I} = \Big(R_{jk}(n)n^{(d_j+d_k)-1}\Big)_{j,k=1,\ldots,p}, \tag{2.1}$$

where $D = \mathrm{diag}(d_1, \ldots, d_p)$ with $d_j \in (0, 1/2), j = 1, \ldots, p$, and $R(u) = (R_{jk}(u))_{j,k=1,\ldots,p}$ is an $\mathbb{R}^{p\times p}$-valued function satisfying

$$R(u) \sim R = (R_{jk})_{j,k=1,\ldots,p}, \quad \text{as } u \to +\infty, \tag{2.2}$$

for some $p \times p$ matrix $R$, where $R_{jk} \in \mathbb{R}$.

**Definition 2.2** (*Spectral domain*) A multivariate ($p$-vector) second-order stationary time series is LRD if its spectral density matrix function satisfies

$$f(\lambda) = \lambda^{-D}G(\lambda)\lambda^{-D^*} = \Big(G_{jk}(\lambda)\lambda^{-(d_j+d_k)}\Big)_{j,k=1,\ldots,p}, \tag{2.3}$$

where $D = \mathrm{diag}(d_1, \ldots, d_p)$ with $d_j \in (0, 1/2), j = 1, \ldots, p$, and $G(\lambda) = (G_{jk}(\lambda))_{j,k=1,\ldots,p}$ is a $\mathbb{C}^{p\times p}$-valued, Hermitian symmetric, non-negative definite matrix function satisfying

$$G(\lambda) \sim G = (G_{jk})_{j,k=1,\ldots,p} = \Big(g_{jk}e^{i\phi_{jk}}\Big)_{j,k=1,\ldots,p}, \qquad \text{as } \lambda \downarrow 0, \tag{2.4}$$

for some $p\times p$, Hermitian symmetric, non-negative definite matrix $G$, where $g_{jk} \in \mathbb{R}$, $\phi_{jk} \in (-\pi, \pi]$.

A number of comments regarding Definitions 2.1 and 2.2 are in place. First, note that the component series $\{X_n^j\}_{n\in\mathbb{Z}}$, $j = 1, \ldots, p$, of a multivariate LRD series are LRD with parameters $d_j$, $j = 1, \ldots, p$. (Note that $\phi_{jj} = 0$ since the matrix $G$ is Hermitian symmetric and hence has real-valued entries on the diagonal.) Another possibility would be to require that at least one of the component series $\{X_n^j\}_{n\in\mathbb{Z}}$ is LRD. This could be achieved by assuming in (2.3) that $d_j \in [0, 1/2)$, $j = 1, \ldots, p$, and that at least one $d_j > 0$. But note that such assumption is not appropriate in (2.1).

Second, note that the structure of (2.3) is such that $f(\lambda)$ is Hermitian and non-negative definite. $D^*$ appearing in (2.3) can be replaced by $D$ since it is diagonal and consists of real-valued entries. We write $D^*$ to make the non-negative definiteness of $f(\lambda)$ more evident. The entries $\phi_{jk}$ in (2.4) are referred to as *phase parameters*. Their role will become more apparent in Proposition 2.1 below where Definitions 2.1 and 2.2 are compared.

There are two common but slightly different ways to represent the phase parameters. Note that a complex number $z = z_1 + iz_2 \in \mathbb{C}$, $z_1, z_2 \in \mathbb{R}$, can be represented as (assuming $z_1 \neq 0$ in the second relation below and denoting the principle value of the argument by $\mathrm{Arg}(z) \in (-\pi, \pi]$)

$$\begin{aligned} z &= |z|e^{i\mathrm{Arg}(z)} && (2.5)\\ &= \sqrt{z_1^2+z_2^2}e^{i(\arctan(\frac{z_2}{z_1})+\pi\mathrm{sign}(z_2)1_{\{z_1<0\}})} = \mathrm{sign}(z_1)\sqrt{z_1^2+z_2^2}e^{i\arctan(\frac{z_2}{z_1})}\\ &= z_1\sqrt{1+\frac{z_2^2}{z_1^2}}e^{i\arctan(\frac{z_2}{z_1})} = z_1\sqrt{1+\tan^2(\arctan\frac{z_2}{z_1})}e^{i\arctan(\frac{z_2}{z_1})}\\ &= \frac{z_1}{\cos\phi}e^{i\phi}, \quad \text{with} \quad \phi = \arctan(\frac{z_2}{z_1}). && (2.6) \end{aligned}$$

The two specifications of $g_{jk}$ and the phase $\phi_{jk}$ correspond to (2.5) (e.g. Brockwell and Davis (2009), p. 422) and (2.6) (e.g. Hannan (1970), pp. 43–44):

$$g_{jk} = |G_{jk}|, \quad \phi_{jk} = \mathrm{Arg}(G_{jk}), \tag{2.7}$$

$$g_{jk} = \frac{\Re G_{jk}}{\cos\phi_{jk}}, \quad \phi_{jk} = \arctan\frac{\Im G_{jk}}{\Re G_{jk}}. \tag{2.8}$$

Note that, in the case (2.8), $\phi_{jk} \in (-\pi/2, \pi/2)$.

It should also be noted that the phase parameters are unique to the LRD case. Taking

$$\sum_{n=-\infty}^{\infty} |\gamma(n)| < \infty \tag{2.9}$$

for the definition of *short-range dependent* (SRD) series, we have

$$f(\lambda) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-in\lambda}\gamma(n). \tag{2.10}$$

In particular, $f(0) = (2\pi)^{-1}\sum_{n=-\infty}^{\infty}\gamma(n)$ consists of real-valued entries, and

$$f(\lambda) \sim G \ (= f(0)), \quad \text{as } \lambda \to 0, \tag{2.11}$$

where $G$ consists of real-valued entries. The relation (2.11) corresponds to (2.3)$-$(2.4) with $d_j = 0$, $j = 1, \dots, p$, and all phase parameters $\phi_{jk} = 0$.

Third, the squared coherence function

$$\mathcal{H}_{jk}^2(\lambda) = \frac{|f_{jk}(\lambda)|^2}{f_{jj}(\lambda)f_{kk}(\lambda)}$$

satisfies $0 \leq \mathcal{H}_{jk}^2(\lambda) \leq 1$. As $\lambda \to 0$, this translates into

$$0 \leq \lim_{\lambda\to 0}\frac{|G_{jk}|^2\lambda^{-2(d_j+d_k)}}{G_{jj}\lambda^{-2d_j}G_{kk}\lambda^{-2d_k}} = \frac{|G_{jk}|^2}{G_{jj}G_{kk}} \leq 1 \tag{2.12}$$

and also explains why the choice of $\lambda^{-(d_j+d_k)}$ is natural for the cross-spectral density $f_{jk}(\lambda)$.

Fourth, note also that (2.3) is considered for $\lambda > 0$, $\lambda \to 0$. Since $f$ is Hermitian symmetric, (2.3)−(2.4) can be replaced by

$$f_{jk}(\lambda) = G_{jk}(\lambda)|\lambda|^{-(d_j+d_k)} \sim g_{jk}e^{i\phi_{jk}\mathrm{sign}(\lambda)}|\lambda|^{-(d_j+d_k)}, \quad \text{as } \lambda \to 0, \tag{2.13}$$

where $G_{jk}(-\lambda) = G_{jk}(\lambda)^*$, if both positive and negative $\lambda$'s are considered. (Note that since $G$ is Hermitian symmetric, we have $g_{jk} = g_{kj}$.)

Proposition 2.1 below compares Definitions 2.1 and 2.2. It uses the notion of a quasi-monotone slowly varying function whose definition is recalled in Appendix A. The proof of the proposition can also be found in the appendix. As usual, $\Gamma(\cdot)$ denotes the gamma function.

**Proposition 2.1** *(i) Suppose the component functions $R_{jk}$ are quasi-monotone slowly varying. Then, Definition 2.1 implies Definition 2.2 with*

$$G_{jk} = \frac{\Gamma(d_j+d_k)}{2\pi}\Big\{(R_{jk}+R_{kj})\cos(\frac{\pi}{2}(d_j+d_k)) - i(R_{jk}-R_{kj})\sin(\frac{\pi}{2}(d_j+d_k))\Big\} \tag{2.14}$$

*in the relation (2.4). In the specification (2.8), for example, $G_{jk} = g_{jk}e^{i\phi_{jk}}$ with*

$$\phi_{jk} = -\arctan\Big\{\frac{R_{jk}-R_{kj}}{R_{jk}+R_{kj}}\tan(\frac{\pi}{2}(d_j+d_k))\Big\}, \tag{2.15}$$

$$g_{jk} = \frac{\Gamma(d_j+d_k)(R_{jk}+R_{kj})\cos(\frac{\pi}{2}(d_j+d_k))}{2\pi\cos(\phi_{jk})}. \tag{2.16}$$

*(ii) Suppose the component functions $\Re G_{jk}$, $\Im G_{jk}$ are quasi-monotone slowly varying. Then, Definition 2.2 implies Definition 2.1 with*

$$R_{jk} = 2\Gamma(1-(d_j+d_k))\Big\{\Re G_{jk}\sin(\frac{\pi}{2}(d_j+d_k)) - \Im G_{jk}\cos(\frac{\pi}{2}(d_j+d_k))\Big\} \tag{2.17}$$

*in the relation (2.2).*

The relation (2.15) sheds light on the phase parameters $\phi_{jk}$. Note that $\phi_{jk} = 0$ if and only if $R_{jk} = R_{kj}$. In view of (2.1)–(2.2), the last property corresponds to $\gamma_{jk}(n)$ being symmetric as $n \to \infty$ and $n \to -\infty$, that is, $\gamma_{jk}(-n) \sim \gamma_{jk}(n)$, as $n \to \infty$. (We used here the fact that $\gamma(-n) = \gamma(n)'$ and hence $\gamma_{jk}(-n) = \gamma_{kj}(n)$.)

# 3 Non-causal linear representations

We are interested here in linear representations (1.9) of multivariate LRD time series. In the next result, we show that linear time series (1.9) with power-law coefficients $\Psi_k$ in the asymptotic sense as $k \to \infty$ and $k \to -\infty$, are multivariate LRD. We argue at the end of the section that such non-causal time series can lead to general phase parameters.

**Proposition 3.1** *Let $\{\epsilon_n\}_{n\in\mathbb{Z}}$ be an $\mathbb{R}^p$-valued white noise, satisfying $\mathbb{E}\epsilon_n = 0$ and $\mathbb{E}\epsilon_n\epsilon_n' = I$. Let also $\{\Psi_m = (\psi_{jk,m})_{j,k=1,\dots,p}\}_{m\in\mathbb{Z}}$ be a sequence of real-valued matrices such that*

$$\psi_{jk,m} = L_{jk}(m)|m|^{d_j-1}, \quad m \in \mathbb{Z}, \tag{3.1}$$

*where $d_j \in (0, 1/2)$ and $L(m) = \big(L_{jk}(m)\big)_{j,k=1,\dots,p}$ is an $\mathbb{R}^{p\times p}$-valued function satisfying*

$$L(m) \sim A^+, \quad \textit{as} \quad m \to \infty, \quad \textit{and} \quad L(m) \sim A^-, \quad \textit{as} \quad m \to -\infty, \tag{3.2}$$

*for some $p \times p$ real-valued matrices $A^+ = (\alpha^+_{jk})_{j,k=1,\dots,p}$, $A^- = (\alpha^-_{jk})_{j,k=1,\dots,p}$. Then, the time series $X_n$ given by a linear representation*

$$X_n = \sum_{m=-\infty}^{\infty} \Psi_m \epsilon_{n-m}, \tag{3.3}$$

*is LRD in the sense of Definition 2.1 with*

$$R_{jk} = \frac{\Gamma(d_j)\Gamma(d_k)}{\Gamma(d_j+d_k)} \left( c^1_{jk} \frac{\sin(\pi d_j)}{\sin(\pi(d_j+d_k))} + c^2_{jk} + c^3_{jk} \frac{\sin(\pi d_k)}{\sin(\pi(d_j+d_k))} \right), \tag{3.4}$$

*where*

$$\begin{array}{lllll} c^1_{jk} & = & \sum_{t=1}^p \alpha^-_{jt}\alpha^-_{kt} & = & (A^-(A^-)^*)_{jk}, \\ c^2_{jk} & = & \sum_{t=1}^p \alpha^-_{jt}\alpha^+_{kt} & = & (A^-(A^+)^*)_{jk}, \\ c^3_{jk} & = & \sum_{t=1}^p \alpha^+_{jt}\alpha^+_{kt} & = & (A^+(A^+)^*)_{jk}. \end{array} \tag{3.5}$$

The proof of Proposition 3.1 can be found in Appendix A. Entries of the matrices $A^+$ and $A^-$ in (3.2) are allowed to be zero. In particular, the case

$$\psi_{jk,m} = l_{jk}(m)|m|^{d_{jk}-1}, \quad m \in \mathbb{Z}, \tag{3.6}$$

with possibly different $d_{jk} \in (0,1/2)$ across $k$ for fixed $j$ and $l_{jk}(m) \sim \beta^{\pm}_{jk}$ with $\beta^{\pm}_{jk} \in \mathbb{R}$, as $m \to \pm\infty$, can be expressed as (3.1) with $d_j = \max_k d_{jk}$.

Note also that the time series (3.3) is proved to be LRD in the sense of Definition 2.1. In view of Proposition 2.1, $(i)$, the time series is also expected to be LRD in the sense of Definition 2.2 with $G_{jk}$ given by (2.14). To calculate $G_{jk}$, note that

$$\begin{aligned} R_{jk} + R_{kj} &= \frac{\Gamma(d_j)\Gamma(d_k)}{\Gamma(d_j+d_k)} \left( c^1_{jk} \frac{\sin(\pi d_j) + \sin(\pi d_k)}{\sin(\pi(d_j+d_k))} + c^2_{jk} + c^2_{kj} + c^3_{jk} \frac{\sin(\pi d_k) + \sin(\pi d_j)}{\sin(\pi(d_j+d_k))} \right) \\ &= \frac{\Gamma(d_j)\Gamma(d_k)}{\Gamma(d_j+d_k)} \left( c^1_{jk} \frac{\cos(\frac{\pi}{2}(d_j-d_k))}{\cos(\frac{\pi}{2}(d_j+d_k))} + c^2_{jk} + c^2_{kj} + c^3_{jk} \frac{\cos(\frac{\pi}{2}(d_j-d_k))}{\cos(\frac{\pi}{2}(d_j+d_k))} \right), \end{aligned}$$

where we used basic trigonometric identities and the facts that $c^1_{jk} = c^1_{kj}$, $c^3_{jk} = c^3_{kj}$. Hence,

$$\begin{aligned} &\Gamma(d_j+d_k)(R_{jk}+R_{kj})\cos(\frac{\pi}{2}(d_j+d_k)) \\ &\qquad = \Gamma(d_j)\Gamma(d_k) \left( (c^1_{jk}+c^3_{jk})\cos(\frac{\pi}{2}(d_j-d_k)) + (c^2_{jk}+c^2_{kj})\cos(\frac{\pi}{2}(d_j+d_k)) \right). \end{aligned} \tag{3.7}$$

Similarly, one can show that

$$\begin{aligned} &\Gamma(d_j+d_k)(R_{jk}-R_{kj})\sin(\frac{\pi}{2}(d_j+d_k)) \\ &\qquad = \Gamma(d_j)\Gamma(d_k) \left( (c^1_{jk}-c^3_{jk})\sin(\frac{\pi}{2}(d_j-d_k)) + (c^2_{jk}-c^2_{kj})\sin(\frac{\pi}{2}(d_j+d_k)) \right). \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), the relation (2.14) now yields

$$\begin{aligned} G_{jk} = \frac{\Gamma(d_j)\Gamma(d_k)}{\Gamma(d_j+d_k)} &\Big( (c^1_{jk}+c^3_{jk})\cos(\frac{\pi}{2}(d_j-d_k)) + (c^2_{jk}+c^2_{kj})\cos(\frac{\pi}{2}(d_j+d_k)) \\ &\quad + i(c^1_{jk}-c^3_{jk})\sin(\frac{\pi}{2}(d_j-d_k)) + i(c^2_{jk}-c^2_{kj})\sin(\frac{\pi}{2}(d_j+d_k)) \Big) \\ = \frac{\Gamma(d_j)\Gamma(d_k)}{\Gamma(d_j+d_k)} &\left( c^1_{jk} e^{i\frac{\pi}{2}(d_j-d_k)} + c^3_{jk} e^{-i\frac{\pi}{2}(d_j-d_k)} + c^2_{jk} e^{i\frac{\pi}{2}(d_j+d_k)} + c^2_{kj} e^{-i\frac{\pi}{2}(d_j+d_k)} \right). \end{aligned} \tag{3.9}$$

Setting

$$F = \operatorname{diag}\left(\Gamma(d_1)e^{i\frac{\pi}{2}d_1}, \dots, \Gamma(d_p)e^{i\frac{\pi}{2}d_p}\right) \tag{3.10}$$

and using (3.5), the relation (3.9) can also be expressed as

$$\begin{aligned} G_{jk} &= \frac{\Gamma(d_j)\Gamma(d_k)}{\Gamma(d_j+d_k)} \Big( (A^-(A^-)^*)_{jk} e^{i\frac{\pi}{2}(d_j-d_k)} + (A^+(A^+)^*)_{jk} e^{-i\frac{\pi}{2}(d_j-d_k)} \\ &\qquad + (A^-(A^+)^*)_{jk} e^{i\frac{\pi}{2}(d_j+d_k)} + (A^+(A^-)^*)_{jk} e^{-i\frac{\pi}{2}(d_j+d_k)} \Big) \\ &= \frac{1}{2\pi} \left( (FA^-)(FA^-)^* + (F^*A^+)(F^*A^+)^* + (FA^-)(F^*A^+)^* + (F^*A^+)(FA^-)^* \right)_{jk} \\ &= \frac{1}{2\pi} \left( (F^*A^+ + FA^-)(F^*A^+ + FA^-)^* \right)_{jk}. \end{aligned} \tag{3.11}$$

The relation (3.11) can also be derived informally as follows. It is expected that the time series (3.3) has spectral density

$$\frac{1}{2\pi} \left( \sum_{m=-\infty}^{\infty} \Psi_m e^{-im\lambda} \right) \left( \sum_{m=-\infty}^{\infty} \Psi_m^* e^{im\lambda} \right). \tag{3.12}$$

Be Lemma A.1 in Appendix A, it is expected that

$$\sum_{m=-\infty}^{\infty} \Psi_m e^{-im\lambda} = \sum_{m=1}^{\infty} \Psi_m e^{-im\lambda} + \sum_{m=0}^{\infty} \Psi_{-m} e^{im\lambda} \sim \lambda^{-D} \left( F^*A^+ + FA^- \right), \tag{3.13}$$

which, when combined with (3.12), is consistent with (3.11).

Finally, note that, for fixed $d_j$'s and $G = (G_{jk})_{j,k=1,\dots,p}$, we can find matrices $A^+$, $A^-$, so that (3.11) holds. Indeed, since $G$ is Hermitian symmetric and non-negative definite, we have $G = WW^*$ for some matrix $W$. The real matrices $A^+$, $A^-$ can now be found by setting $(2\pi)^{-1}(F^*A^+ + FA^-) = W$. (Note that, since $e^{-i\pi d/2}$ and $e^{i\pi d/2}$ are linearly independent, there are real $\alpha^+$ and $\alpha^-$ such that $e^{-i\pi d/2}\alpha^+ + e^{i\pi d/2}\alpha^- = y$ for any $y \in \mathbb{C}$.) In particular, any phase $\phi_{jk}$ can be obtained with a suitable choice of $A^+$, $A^-$.

# 4 Causal linear representations

In this section, we focus on causal linear representations of multivariate LRD series, that is, the representations (1.9) with $\Psi_k = 0$ for $k < 0$. As shown in Section 4.1 below, causal representations with power-law coefficients can only have very special phase parameters. Causal representations with zero and more general phases, based on trigonometric power-law coefficients (1.10), are considered in Sections 4.2 and 4.3.

## 4.1 The special case of power-law coefficients

One consequence of the results of Section 3 is that the causal representations of multivariate LRD series with power-law coefficients can only have very special phase parameters. The next result restates Proposition 3.1 in the causal case.

**Corollary 4.1** *Let $\{\epsilon_n\}_{n\in\mathbb{Z}}$ and $\{\Psi_m\}_{m\in\mathbb{Z}}$ be as in Proposition 3.1 but with $\Psi_m = 0$ for $m < 0$. Then, the time series $X_n$ given by a causal linear representation*

$$X_n = \sum_{m=0}^{\infty} \Psi_m \epsilon_{n-m}, \tag{4.1}$$

*is LRD in the sense of Definition 2.1 with*

$$R_{jk} = \frac{\Gamma(d_j)\Gamma(d_k)}{\Gamma(d_j+d_k)} \frac{\sin(\pi d_k)}{\sin(\pi(d_j+d_k))} (A^+(A^+)^*)_{jk}. \tag{4.2}$$

Arguing as for (3.11), the relation (4.2) yields

$$G_{jk} = \frac{\Gamma(d_j)\Gamma(d_k)}{2\pi} (A^+(A^+)^*)_{jk} e^{-i\frac{\pi}{2}(d_j-d_k)} \tag{4.3}$$

and hence the phase parameters

$$\phi_{jk} = -\frac{\pi}{2}(d_j - d_k). \tag{4.4}$$

This can also be deduced from (3.12)–(3.13).

### 4.2 The case of zero phases

The causal time series (4.1) with power-law coefficients leads necessarily to the phase parameters (4.4). What coefficient matrices $\Psi_k$ could one take to obtain general phases? (Such coefficient matrices exist in theory by the multivariate version of the Paley-Wiener theorem.) It is instructive to begin the discussion with the case of zero phases (that is, the symmetric case), before moving to the general case.

Informally, in the case of zero phases, we are looking for coefficient matrices $\Psi_m$ such that, as $\lambda \downarrow 0$,

$$\left(\sum_{m=0}^{\infty} \Psi_m e^{-im\lambda}\right)\left(\sum_{m=0}^{\infty} \Psi_m^* e^{im\lambda}\right) \sim \lambda^{-D} G \lambda^{-D^*}, \tag{4.5}$$

where $G$ is real-valued (and we included the factor $1/2\pi$ on the left-hand side of (3.12) into $\Psi_m$'s). This relation would follow from

$$\left(\sum_{m=0}^{\infty} \psi_{jk,m} e^{-im\lambda}\right)\left(\sum_{m=0}^{\infty} \psi_{j'k',m} e^{im\lambda}\right) \sim c_{jk,j'k'} \lambda^{-(d_j+d_{j'})} \tag{4.6}$$

with real $c_{jk,j'k'}$. This in turn would suggest to look for coefficients $\psi_m$ such that

$$\sum_{m=0}^{\infty} \psi_m e^{-im\lambda} \sim c\lambda^{-d} \tag{4.7}$$

with real $c$. Note, however, that (4.7) with real $c$ is not plausible. Writing $c\lambda^{-d} = ci^d(i\lambda)^{-d}$ the behavior of $(i\lambda)^{-d}$ can be captured by taking power-law coefficients. It is, however, impossible to recover $i^d$ through a Fourier series of real coefficients. In fact, (4.7) is only expected with complex $c = Ci^{-d}$, corresponding to power-law coefficients $\psi_m$. But again, power-law coefficients lead to zero phase only when the component LRD parameters are all identical.

Instead of (4.7), another possibility is to look for coefficients $\psi_m$ such that

$$\sum_{m=0}^{\infty} \psi_m e^{-im\lambda} \sim c\lambda^{-d} e^{ia(\lambda)}, \tag{4.8}$$

where $a(\lambda) \to \infty$, as $\lambda \downarrow 0$. Moreover, $a(\lambda)$ should be flexible enough in its relation to $d$. The idea here is that the complex-valued terms $e^{ia(\lambda)}$ associated with the two Fourier series would cancel out on the left-hand side of (4.6). In fact, coefficients whose Fourier transform behaves as (4.8) have already been studied by Wainger (1965).

For example, adapting the arguments of Wainger (1965), we show in Appendix B that

$$\sum_{n=0}^{\infty} n^{-b} \cos(2\pi n^a) e^{2\pi i n x} \sim c_{a,b} x^{-d} e^{-i(\xi_a x^{-\frac{a}{1-a}} + \psi)}, \tag{4.9}$$

as $x \downarrow 0$, where $0 < a < 1$, $0 < b \leq 1 - \frac{a}{2}$, $\psi = -\frac{\pi}{4}$, $c_{a,b}$ and $\xi_a$ are two non-zero real constants and

$$d = \frac{1 - b - \frac{a}{2}}{1 - a}. \tag{4.10}$$

The next proposition, Proposition 4.1, uses (4.9) to construct multivariate LRD series with zero phases. Before stating the proposition, we shed some light on (4.9)–(4.10) and one further assumption to be made.

Note that, viewing $d$ in (4.10) as a LRD parameter, we need $d < 1/2$ which translates to

$$\frac{1}{2} < b. \tag{4.11}$$

This additional assumption will be made in the proposition below. Under (4.11), the coefficients $n^{-b}\cos(2\pi n^a)$ are also square-summable and thus can be used to define linear time series. When (4.11) holds, observe that

$$d = \frac{1 - b - \frac{a}{2}}{1 - a} < 1 - b =: d_0, \tag{4.12}$$

where $d_0$ corresponds to the value of $d$ in (4.10) when formally setting $a = 0$ in the left-hand side of (4.9) (see Lemma A.1). Moreover, when $a_1 < a_2$ (and (4.11) holds),

$$\frac{1 - b - \frac{a_1}{2}}{1 - a_1} > \frac{1 - b - \frac{a_2}{2}}{1 - a_2}. \tag{4.13}$$

Thus, viewing $d$ as a LRD parameter, it decreases from $d_0 = 1 - b$ associated with power-law coefficients as $a$ increases.

**Proposition 4.1** *Let $\{\epsilon_n\}_{n\in\mathbb{Z}}$ be an $\mathbb{R}^p$-valued white noise, satisfying $\mathbb{E}\epsilon_n = 0$ and $\mathbb{E}\epsilon_n\epsilon_n' = I$. Let also $\{\Psi_m = (\psi_{jk,m})_{j,k=1,\ldots,p}\}_{m\geq 0}$ be a sequence of real-valued matrices such that*

$$\psi_{jk,m} = \alpha_{jk} m^{-b_j} \cos(2\pi m^a), \quad m \geq 0, \tag{4.14}$$

*where $\alpha_{jk} \in \mathbb{R}$, $0 < a < 1$, $\frac{1}{2} < b_j \leq 1 - \frac{1}{2}a$, $j = 1, \ldots, p$. Then, the time series*

$$X_n = \sum_{m=0}^{\infty} \Psi_m \epsilon_{n-m}, \tag{4.15}$$

*is LRD in the sense of Definition 2.2 with*

$$d_j = \frac{1 - b_j - \frac{a}{2}}{1 - a}, \tag{4.16}$$

$$G_{jk} = (2\pi)^{d_j + d_k - 1} c_{a,b_j} c_{a,b_k} (AA^*)_{jk}, \tag{4.17}$$

*where* $A = (\alpha_{jk})_{j,k=1,\dots,p}$*,* $c_{a,b}$ *is a non-zero real constant given in Theorem B.1, and hence the phase parameters*

$$\phi_{jk} = 0.$$

*When the cosine in (4.14) is replaced by sine, the statements above continue to hold but with*

$$G_{jk} = -(2\pi)^{d_j + d_k - 1} c_{a,b_j} c_{a,b_k} (AA^*)_{jk}. \tag{4.18}$$

Proposition 4.1 is proved in Appendix A.

### 4.3 The case of general phases

We showed in Proposition 4.1 that trigonometric power-law coefficients lead to zero phases for multivariate LRD series. The next result shows that linear combinations of trigonometric power-law coefficients can capture general phases (see also the discussion following Proposition 4.2).

**Proposition 4.2** *Let* $\{\epsilon_n\}_{n\in\mathbb{Z}}$ *be as in Proposition 4.1. Let also* $\{\Psi_m = (\psi_{jk,m})_{j,k=1,\dots,p}\}_{m\in\mathbb{Z}}$ *be a sequence of real-valued matrices such that*

$$\psi_{jk,m} = \alpha_{jk} m^{-b_j} \cos(2\pi m^a) + \beta_{jk} m^{-b_j} \sin(2\pi m^a), \quad m \geq 0, \tag{4.19}$$

*where* $\alpha_{jk}, \beta_{jk} \in \mathbb{R}$*,* $0 < a < 1$*,* $\frac{1}{2} < b_j \leq 1 - \frac{1}{2}a$*,* $j = 1, \dots, p$*. Then, the time series*

$$X_n = \sum_{m=0}^{\infty} \Psi_m \epsilon_{n-m}, \tag{4.20}$$

*is LRD in the sense of Definition 2.2 with*

$$d_j = \frac{1 - b_j - \frac{a}{2}}{1 - a}, \tag{4.21}$$

$$G_{jk} = (2\pi)^{d_j + d_k - 1} c_{a,b_j} c_{a,b_k} \sum_{t=1}^{p} \overline{z}_{jt} z_{kt}, \tag{4.22}$$

*where* $z_{jk} = \alpha_{jk} + i\beta_{jk}$*, and* $c_{a,b}$ *is a non-zero real constant given in Theorem B.1.*

Proposition 4.2 is proved in Appendix A.

Note also that the coefficients $\psi_{jk,m}$ in (4.19) can be expressed as (supposing, for example, $\alpha_{jk} \neq 0$)

$$\begin{aligned}
\psi_{jk} &= \alpha_{jk} \left( \cos(2\pi m^a) + \frac{\beta_{jk}}{\alpha_{jk}} \sin(2\pi m^a) \right) m^{-b_j} \\
&= \frac{a_{jk}}{\sqrt{1 + \beta_{jk}^2/\alpha_{jk}^2}} \left( \frac{1}{\sqrt{1 + \beta_{jk}^2/\alpha_{jk}^2}} \cos(2\pi m^a) + \frac{\beta_{jk}/a_{jk}}{\sqrt{1 + \beta_{jk}^2/\alpha_{jk}^2}} \sin(2\pi m^a) \right) m^{-b_j} \\
&= \tau_{jk} \left( \cos(\psi_{jk}) \cos(2\pi m^a) - \sin(\psi_{jk}) \sin(2\pi m^a) \right) m^{-b_j} \\
&= \tau_{jk} m^{-b_j} \cos(2\pi m^a + \psi_{jk}),
\end{aligned} \tag{4.23}$$

where $\tau_{jk} = a_{jk}/\sqrt{1+\beta_{jk}^2/\alpha_{jk}^2}$ and $\psi_{jk} \in (-\pi/2, \pi/2)$ is such that $\tan(\psi_{jk}) = -\beta_{jk}/\alpha_{jk}$.

Note that the matrix $G = (G_{jk})_{j,k=1,\dots,p}$ with the entries (4.22) can be written as

$$G = ZZ^*,$$

where $Z = ((2\pi)^{d_j} c_{a,b_j} \overline{z}_{jk})_{j,k=1,\dots,p}$. Any Hermitian symmetric, non-negative definite $G$ can be written as $G = WW^*$ and $\alpha_{jk}, \beta_{jk}$ can be found by setting $Z = W$.

# 5 Multivariate FARIMA$(0, D, 0)$ series

We provide here a multivariate, non-causal extension of FARIMA series. Let $D = \operatorname{diag}(d_1, \dots, d_p)$ with $d_j < 1/2$, $j = 1, \dots, p$, and $Q_+ = (q_{jk}^+)$, $Q_- = (q_{jk}^-) \in \mathbb{R}^{p\times p}$. Let also $\{\epsilon_n\}_{n\in\mathbb{Z}}$ be an $\mathbb{R}^p$-valued white noise series, satisfying $\mathbb{E}\epsilon_n = 0$ and $\mathbb{E}\epsilon_n\epsilon_n' = I$. Define a *multivariate FARIMA$(0, D, 0)$ series* as

$$X_n = (I-B)^{-D}Q_+\epsilon_n \ + \ (I-B^{-1})^{-D}Q_-\epsilon_n, \tag{5.1}$$

where $B$ is the backshift operator. The series $X_n$ is thus given by a non-causal linear representation (when $Q_-$ is not identically zero). In the next result, we give the exact form of the autocovariance matrix function of the multivariate FARIMA$(0, D, 0)$ series in (5.1).

**Proposition 5.1** *The $(j, k)$ component $\gamma_{jk}(n)$ of the autocovariance matrix function $\gamma(n)$ of the multivariate FARIMA$(0, D, 0)$ series in (5.1) is given by*

$$\gamma_{jk}(n) = \frac{1}{2\pi}\left(b_{jk}^1\gamma_{1,jk}(n) + b_{jk}^2\gamma_{2,jk}(n) + b_{jk}^3\gamma_{3,jk}(n) + b_{jk}^4\gamma_{4,jk}(n)\right), \tag{5.2}$$

*where*

$$\begin{aligned} b_{jk}^1 &= \sum_{t=1}^p q_{jt}^- q_{jt}^- = (Q^-(Q^-)^*)_{jk}, \quad & b_{jk}^3 &= \sum_{t=1}^p q_{jt}^+ q_{jt}^+ = (Q^+(Q^+)^*)_{jk}, \\ b_{jk}^2 &= \sum_{t=1}^p q_{jt}^- q_{jt}^+ = (Q^-(Q^+)^*)_{jk}, \quad & b_{jk}^4 &= \sum_{t=1}^p q_{jt}^+ q_{jt}^- = (Q^+(Q^-)^*)_{jk}, \end{aligned} \tag{5.3}$$

*and*

$$\begin{aligned} \gamma_{1,jk}(n) &= \gamma_{3,kj}(n) = 2\Gamma(1-d_j-d_k)\sin(\pi d_k)\tfrac{\Gamma(n+d_k)}{\Gamma(n+1-d_j)}, \\ \gamma_{4,jk}(n) &= \gamma_{2,jk}(-n) = \begin{cases} 2\pi\frac{1}{\Gamma(d_j+d_k)}\frac{\Gamma(d_j+d_k+n)}{\Gamma(1+n)} &, \ n = 0,\ 1, 2, \dots, \\ 0 &, \ n = -1,\ -2, \dots \end{cases} \end{aligned} \tag{5.4}$$

**Remark 5.1** Since $(I-B)^{-d} = \sum_{j=0}^\infty b_j B^j$ with $b_j = j^{d-1}/\Gamma(d)$, as $j \to \infty$, observe that the FARIMA$(0, D, 0)$ series satisfies (3.1) with

$$A^+ = \Gamma(D)^{-1}Q^+, \quad A^- = \Gamma(D)^{-1}Q^-, \tag{5.5}$$

where $\Gamma(D)^{-1} = \operatorname{diag}(\Gamma(d_1)^{-1}, \dots, \Gamma(d_p)^{-1})$. By using $\Gamma(j+a)/\Gamma(j+b) \sim j^{a-b}$, as $j \to \infty$, observe also that, as $n \to \infty$,

$$\gamma_{1,jk}(n) \sim 2\Gamma(1-d_j-d_k)\sin(\pi d_j)\frac{\Gamma(n+d_j)}{\Gamma(n+1-d_k)} = \frac{2\pi\sin(\pi d_j)}{\Gamma(d_j+d_k)\sin(\pi(d_j+d_k))}n^{d_j+d_k-1}, \tag{5.6}$$

$$\gamma_{4,jk}(n) \sim \frac{2\pi}{\Gamma(d_j+d_k)}n^{d_j+d_k-1}. \tag{5.7}$$

The relations (5.5)–(5.7) show that (5.2)–(5.4) are consistent with (3.4)–(3.5).

PROOF: By using Theorem 11.8.3 in Brockwell and Davis (2009), the FARIMA$(0, D, 0)$ series in (5.1) has the spectral density matrix

$$f(\lambda) = \frac{1}{2\pi} G(\lambda) G(\lambda)^*, \tag{5.8}$$

where $G(\lambda) = (1 - e^{-i\lambda})^{-D} Q_+ \; + \; (1 - e^{i\lambda})^{-D} Q_-$. This can be expressed component-wise as

$$f_{jk}(\lambda) = \frac{1}{2\pi} g_j(\lambda) g_k(\lambda)^*, \tag{5.9}$$

where $g_j$ is the $j$th row of $G$. Then, the $(j, k)$ component of the autocovariance matrix is

$$\begin{aligned} \gamma_{jk}(n) &= \int_0^{2\pi} e^{in\lambda} f_{jk}(\lambda) d\lambda = \frac{\sigma^2}{2\pi} \int_0^{2\pi} e^{in\lambda} g_j(\lambda) g_k(\lambda)^* d\lambda \\ &= \frac{1}{2\pi} \left( b^1_{jk} \gamma_{1,jk}(n) + b^2_{jk} \gamma_{2,jk}(n) + b^3_{jk} \gamma_{3,jk}(n) + b^4_{jk} \gamma_{4,jk}(n) \right), \end{aligned} \tag{5.10}$$

where $b^1_{jk}, b^2_{jk}, b^3_{jk}, b^4_{jk}$ are given in (5.3), and

$$\gamma_{1,jk}(n) = \gamma_{3,kj}(n) = \int_0^{2\pi} e^{in\lambda} (1 - e^{i\lambda})^{-d_j} (1 - e^{-i\lambda})^{-d_k} d\lambda,$$

$$\gamma_{2,jk}(n) = \int_0^{2\pi} e^{in\lambda} (1 - e^{i\lambda})^{-(d_j + d_k)} d\lambda, \qquad \gamma_{4,jk}(n) = \int_0^{2\pi} e^{in\lambda} (1 - e^{-i\lambda})^{-(d_j + d_k)} d\lambda.$$

By writing $1 - e^{\pm i\lambda} = 2 \sin(\frac{\lambda}{2}) e^{\pm i(\lambda - \pi)/2}$, we have

$$\begin{aligned} \gamma_{1,jk}(n) &= \frac{e^{i\pi(d_j - d_k)/2}}{2^{d_j + d_k}} \int_0^{2\pi} e^{in\lambda} \sin^{-d_j - d_k}(\frac{\lambda}{2}) e^{i\lambda(d_k - d_j)/2} d\lambda \\ &= \frac{2e^{i\pi(d_j - d_k)/2}}{2^{d_j + d_k}} \int_0^{\pi} e^{i\omega(2n + d_k - d_j)} \sin^{-d_j - d_k}(\omega) d\omega. \end{aligned}$$

By using Formula 3.892.1 in Gradshteyn and Ryzhik (2007), p. 485, we deduce that

$$\gamma_{1,jk}(n) = \frac{2e^{i\pi(d_j - d_k)/2}}{2^{d_j + d_k}} \frac{\pi e^{i\beta\pi/2}}{2^{\nu - 1} \nu B(\frac{\nu + \beta + 1}{2}, \frac{\nu - \beta + 1}{2})},$$

where $\beta = 2n + d_k - d_j$ and $\nu = 1 - d_k - d_j$. Then,

$$\gamma_{1,jk}(n) = 2\pi(-1)^n \frac{\Gamma(1 - d_j - d_k)}{\Gamma(1 - d_j + n)\Gamma(1 - d_k - n)}. \tag{5.11}$$

Similar calculations yield

$$\gamma_{2,jk}(n) = 2\pi(-1)^n \frac{\Gamma(1 - d_j - d_k)}{\Gamma(1 - n)\Gamma(1 + n - d_j - d_k)}. \tag{5.12}$$

The relations (5.4) can now be deduced from (5.11) and (5.12) by using the identities $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and $\Gamma(z)\Gamma(1 - z) = (-1)^n \Gamma(n + z)\Gamma(1 - n - z)$, $0 < z < 1$. $\Box$

The exact form of the autovovariance function in (5.2)–(5.4) can be used, for example, in a fast generation of the Gaussian FARIMA$(0, D, 0)$ series by using a circulant matrix embedding method (see Helgason et al. (2011)). For example, Figure 1 presents the plots of a bivariate FARIMA$(0, D, 0)$ series with

$$Q^+ = \begin{pmatrix} 0.50246 & 0 \\ 0 & 1.2436 \end{pmatrix}, \quad Q^- = \begin{pmatrix} 0 & -1.8878 \\ 3.4191 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad \phi = 1.4587.$$

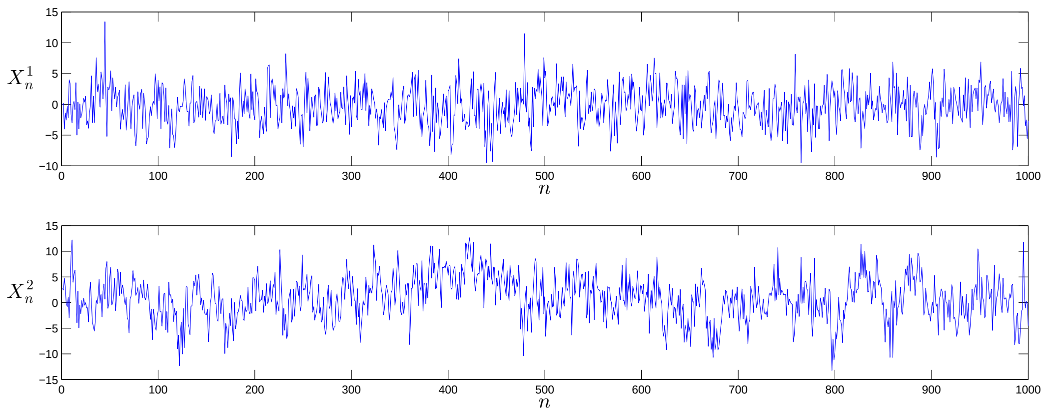


Figure 1: Components of bivariate FARIMA$(0, D, 0)$ series where $D = \mathrm{diag}(0.2, 0.4)$, and $\phi = 1.4587$.

# A Technical proofs

We gather here the proofs of several results given above, starting with Proposition 2.1. Recall that a slowly varying function $L$ is *quasi-monotone* if it is of bounded variation on any compact interval of $[0, \infty)$ and, if for some $\delta > 0$,

$$\int_0^x u^\delta |dL(u)| = O\left(x^\delta L(x)\right), \quad \text{as} \quad x \to \infty \tag{A.1}$$

(Bingham, Goldie and Teugels (1989)). One interest in quasi-monotone slowly varying functions lies in the following classical result (see, for example, Theorem 4.3.2 in Bingham et al. (1989)).

**Lemma A.1** *Suppose $L$ is a quasi-monotone slowly varying function. Let $g(u)$ stand for $\cos(u)$, $\sin(u)$ or $e^{iu}$, and $0 < p < 1$. Then, the following series converges conditionally for all $\lambda \in (0, \pi]$, and*

$$\sum_{k=0}^{\infty} g(k\lambda)\frac{L(k)}{k^p} \sim \lambda^{p-1} L\left(\frac{1}{\lambda}\right)\Gamma(1-p) g\left(\frac{\pi}{2}(1-p)\right), \quad \textit{as} \quad \lambda \to 0. \tag{A.2}$$

A converse of Lemma A.1, allowing one to go from the spectral domain to the time domain, is also available (see (4.3.8) in Bingham et al. (1989)).

**Lemma A.2** *Suppose $l(1/u)$ is a quasi-monotone slowly varying function on $(1/\pi, \infty)$, and $0 < p < 1$. Then,*

$$\int_0^\pi e^{in\lambda}\lambda^{p-1} l(\lambda) d\lambda \sim n^{-p} l\left(\frac{1}{n}\right)\Gamma(p) e^{\frac{i\pi p}{2}}, \quad \textit{as} \quad n \to \infty.$$

Proof of Proposition 2.1: $(i)$ One consequence of Lemma A.1 (omitted here for the shortness sake) is that we can write

$$f_{jk}(\lambda) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-in\lambda}\gamma_{jk}(n). \tag{A.3}$$

(Proving (A.3) amounts to showing that $f_{jk}(\lambda)$ given by the right-hand side of (A.3) satisfies $\int_{-\pi}^{\pi} e^{in\lambda} f_{jk}(\lambda)d\lambda = \gamma_{jk}(n)$.) Then, by using $\gamma_{jk}(-n) = \gamma_{kj}(n)$,

$$f_{jk}(\lambda) = \frac{1}{2\pi}\Big\{ \sum_{n=-\infty}^{\infty} \cos(n\lambda)\gamma_{jk}(n) - i \sum_{n=-\infty}^{\infty} \sin(n\lambda)\gamma_{jk}(n)\Big\}$$

$$= \frac{1}{2\pi}\Big\{\gamma_{jk}(0) + \sum_{n=0}^{\infty} \cos(n\lambda)(\gamma_{jk}(n) + \gamma_{kj}(n))\Big\} - \frac{i}{2\pi}\Big\{ \sum_{n=0}^{\infty} \sin(n\lambda)(\gamma_{jk}(n) - \gamma_{kj}(n))\Big\}$$

$$= \frac{1}{2\pi}\Big\{\gamma_{jk}(0) + \sum_{n=1}^{\infty} \cos(n\lambda)\frac{R_{jk}(n) + R_{kj}(n)}{n^{1-(d_j+d_k)}}\Big\} - \frac{i}{2\pi}\Big\{ \sum_{n=1}^{\infty} \sin(n\lambda)\frac{R_{jk}(n) - R_{kj}(n)}{n^{1-(d_j+d_k)}}\Big\}.$$

It follows from Lemma A.1 that

$$f_{jk}(\lambda) \sim \frac{\Gamma(d_j + d_k)}{2\pi}\lambda^{-(d_j+d_k)}\Big\{(R_{jk} + R_{kj})\cos(\frac{\pi}{2}(d_j + d_k)) - i(R_{jk} - R_{kj})\sin(\frac{\pi}{2}(d_j + d_k))\Big\},$$

showing (2.14). This also immediately yields (2.15)–(2.16) by using (2.8).

$(ii)$ Note that

$$\gamma_{jk}(n) = \int_{-\pi}^{\pi} e^{in\lambda} f_{jk}(\lambda)d\lambda = \int_{-\pi}^{\pi} e^{in\lambda} G_{jk}(\lambda)|\lambda|^{-(d_j+d_k)}d\lambda$$

$$= \int_0^{\pi} e^{in\lambda}(\Re G_{jk}(\lambda) + i\Im G_{jk}(\lambda))\lambda^{-(d_j+d_k)}d\lambda + \int_0^{\pi} e^{-in\lambda}(\Re G_{jk}(-\lambda) + i\Im G_{jk}(-\lambda))\lambda^{-(d_j+d_k)}d\lambda$$

$$= \int_0^{\pi} e^{in\lambda}(\Re G_{jk}(\lambda) + i\Im G_{jk}(\lambda))\lambda^{-(d_j+d_k)}d\lambda + \int_0^{\pi} e^{-in\lambda}(\Re G_{kj}(\lambda) - i\Im G_{kj}(\lambda))\lambda^{-(d_j+d_k)}d\lambda$$

$$= 2\int_0^{\pi} \cos(n\lambda)\Re G_{jk}(\lambda)\lambda^{-(d_j+d_k)}d\lambda - \int_0^{\pi} \sin(n\lambda)\Im G_{jk}(\lambda)\lambda^{-(d_j+d_k)}d\lambda,$$

where we used $\Re G_{jk}(\lambda) = \Re G_{kj}(\lambda)$ and $\Im G_{jk}(\lambda) = -\Im G_{kj}(\lambda)$. By Lemma A.2 we get that

$$\gamma_{jk}(n) \sim 2\Gamma(1 - (d_j + d_k))n^{(d_j+d_k)-1}\Big\{\Re G_{jk}\cos(\frac{\pi}{2}(1 - (d_j + d_k))) - \Im G_{jk}\sin(\frac{\pi}{2}(1 - (d_j + d_k)))\Big\},$$

which yields (2.17). $\square$

We next turn to Proposition 3.1.

Proof of Proposition 3.1: Write the autocovariance function $\gamma(n)$ of $X_n$ as

$$\gamma(n) = \sum_{m=-\infty}^{\infty} \Psi_m\Psi'_{m+n} = \sum_{m=-\infty}^{-n-1} \Psi_m\Psi'_{m+n} + \sum_{m=-n}^{0} \Psi_m\Psi'_{m+n} + \sum_{m=0}^{\infty} \Psi_m\Psi'_{m+n} =: \gamma_1(n)+\gamma_2(n)+\gamma_3(n).$$

Denote by $\gamma_{i,jk}(n)$ the $(j,k)$ component of $\gamma_i(n)$, $i = 1, 2, 3$. Then, by using (3.1), we have

$$\begin{aligned}
\gamma_{1,jk}(n) &= \sum_{m=-\infty}^{-n-1}\sum_{t=1}^{p} L_{jt}(m)L_{kt}(m+n)|m|^{d_j-1}|m+n|^{d_k-1} \\
&= \sum_{t=1}^{p}\sum_{m=n+1}^{\infty} L_{jt}(-m)L_{kt}(n-m)m^{d_j-1}(m+n)^{d_k-1} \\
&= n^{d_j+d_k-1}\sum_{t=1}^{p}\sum_{m=n+1}^{\infty} L_{jt}(-m)L_{kt}(n-m)\left(\frac{m}{n}\right)^{d_j-1}\left(\frac{m}{n}+1\right)^{d_k-1}\frac{1}{n} \\
&\sim n^{d_j+d_k-1}\sum_{t=1}^{p}\alpha^-_{jt}\alpha^-_{kt}\int_1^{\infty} x^{d_j-1}(x-1)^{d_k-1}dx,
\end{aligned}$$

where the last asymptotic relation follows by the dominated convergence theorem and (3.2). By using Formula 3.191.2 in Gradshteyn and Ryzhik (2007), p. 315, we have

$$\gamma_{1,jk}(n) \sim R^1_{jk} n^{d_j+d_k-1}, \quad \text{as} \quad n \to \infty, \tag{A.4}$$

where $R^1_{jk} = \sum_{t=1}^p \alpha^-_{jt}\alpha^-_{kt} \frac{\Gamma(d_k)\Gamma(1-d_j-d_k)}{\Gamma(1-d_j)} n^{d_j+d_k-1}$. Similarly for $\gamma_{2,jk}(n)$ and $\gamma_{3,jk}(n)$, as $n \to \infty$, we have

$$\gamma_{2,jk}(n) \sim n^{d_j+d_k-1} \sum_{t=1}^p \alpha^-_{jt}\alpha^+_{kt} \int_0^1 x^{d_j-1}(1-x)^{d_k-1} dx = R^2_{jk} n^{d_j+d_k-1}, \tag{A.5}$$

$$\gamma_{3,jk}(n) \sim n^{d_j+d_k-1} \sum_{t=1}^p \alpha^+_{jt}a^+_{kt} \int_0^\infty x^{d_j-1}(x+1)^{d_k-1} dx = R^3_{jk} n^{d_j+d_k-1}, \tag{A.6}$$

where $R^2_{jk} = \sum_{t=1}^p \alpha^-_{jt}\alpha^+_{kt} \frac{\Gamma(d_j)\Gamma(d_k)}{\Gamma(d_j+d_k)}$ and $R^3_{jk} = \sum_{t=1}^p \alpha^+_{jt}\alpha^+_{kt} \frac{\Gamma(d_j)\Gamma(1-d_j-d_k)}{\Gamma(1-d_k)}$. Combining (A.4), (A.5) and (A.6), we get (2.1)–(2.2) with

$$R_{jk} = c^1_{jk} \frac{\Gamma(d_k)\Gamma(1-d_j-d_k)}{\Gamma(1-d_j)} + c^2_{jk} \frac{\Gamma(d_j)\Gamma(d_k)}{\Gamma(d_j+d_k)} + c^3_{jk} \frac{\Gamma(d_j)\Gamma(1-d_j-d_k)}{\Gamma(1-d_k)}, \tag{A.7}$$

where $c^1_{jk} = \sum_{t=1}^p \alpha^-_{jt}\alpha^-_{kt}$, $c^2_{jk} = \sum_{t=1}^p \alpha^-_{jt}\alpha^+_{kt}$, $c^3_{jk} = \sum_{t=1}^p \alpha^+_{jt}\alpha^+_{kt}$. The coefficients $R_{jk}$ can be expressed as in (3.4) by using the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, $0 < z < 1$. $\square$

Next, we will prove Propositions 4.1 and 4.2.

Proof of Proposition 4.1: The series $X_n$ in (4.15) is well defined (in the $L^2(\Omega)$-sense) since $b_j > 1/2$ and hence

$$\sum_{m=0}^\infty |\psi_{m,jk}|^2 = \sum_{m=0}^\infty |\alpha_{jk} m^{-b_j} \cos(2\pi m^a)|^2 \le \alpha^2_{jk} \sum_{m=0}^\infty m^{-2b_j} < \infty,$$

for $j, k = 1, \dots, p$. Moreover, from (5.3) in Hannan (1970), p. 61, the spectral density matrix $f(\lambda)$ of $X_n$ is given by

$$f(\lambda) = \frac{1}{2\pi} \left( \sum_{m=0}^\infty \Psi_m e^{-im\lambda} \right) \left( \sum_{m=0}^\infty \Psi_m e^{-im\lambda} \right)^*, \tag{A.8}$$

where the series $\sum_{m=0}^\infty \Psi_m e^{-im\lambda}$ is defined a.e. in the $L^2(-\pi,\pi]$-sense. The $(j,k)$ entry of the series $\sum_{m=0}^\infty \Psi_m e^{-im\lambda}$ is given by $\sum_{m=0}^\infty \psi_{m,jk} e^{-im\lambda} = \alpha_{jk} \sum_{m=0}^\infty m^{-b_j} \cos(2\pi m^a) e^{-im\lambda}$. By Lemma B.3, this entry is equal to $\alpha_{jk} \overline{f_1(\lambda/2\pi)}$, where $f_1(x)$ is defined in Lemma B.1. Hence, with this interpretation and by using Lemma B.1 with $x = \lambda/2\pi$,

$$\sum_{m=0}^\infty \psi_{m,jk} e^{-im\lambda} \sim a_{jk} (2\pi)^{d_j} c_{a,b_j} \lambda^{-d_j} e^{i(\xi_a (2\pi)^{\frac{a}{1-a}} \lambda^{-\frac{a}{1-a}} - \frac{\pi}{4})}, \quad \text{as} \quad \lambda \downarrow 0. \tag{A.9}$$

By using (A.8) and (A.9), we conclude that

$$f_{jk}(\lambda) \sim (2\pi)^{d_j+d_k-1} c_{a,b_j} c_{a,b_k} \sum_{t=1}^p a_{jt} a_{kt} \lambda^{-(d_j+d_k)}. \tag{A.10}$$

Finally, the last statement of the proposition concerning sine can be deduced similarly. □

Proof of Proposition 4.2: As in the proof above, the series $X_n$ in (4.20) is well defined in the $L^2(\Omega)$-sense and its spectral density is given by (A.8), where the series $\sum_{m=0}^{\infty} \Psi_m e^{-im\lambda}$ is defined a.e. in the $L^2(-\pi,\pi]$-sense. The $(j,k)$ entry of the series $\sum_{m=0}^{\infty} \Psi_m e^{-im\lambda}$ is given by $\sum_{m=0}^{\infty} \psi_{m,jk} e^{-im\lambda} = \alpha_{jk} \sum_{m=0}^{\infty} m^{-b_j} \cos(2\pi m^a) e^{-im\lambda} + \beta_{jk} \sum_{m=0}^{\infty} m^{-b_j} \sin(2\pi m^a) e^{-im\lambda}$. By Lemma B.3, this entry is equal to $\alpha_{jk}\overline{f_1(\lambda/2\pi)} + \beta_{jk}\overline{f_2(\lambda/2\pi)}$, where $f_1(x), f_2(x)$ are defined in Lemma B.1. Then, by using Lemma B.1 with $x = \lambda/2\pi$,

$$\sum_{m=0}^{\infty} \psi_{m,jk} e^{-im\lambda} \sim \overline{z}_{jk} (2\pi)^{d_j} c_{a,b_j} \lambda^{-d_j} e^{i(\xi_a (2\pi)^{\frac{a}{1-a}} \lambda^{-\frac{a}{1-a}} - \psi)}, \quad \text{as} \quad \lambda \downarrow 0, \tag{A.11}$$

where $z_{jk} = \alpha_{jk} + i\beta_{jk}$. By using (A.8) and (A.11), we conclude that

$$f_{jk}(\lambda) \sim (2\pi)^{d_j+d_k-1} c_{a,b_j} c_{a,b_k} \sum_{t=1}^{p} \overline{z}_{jt} z_{kt} \lambda^{-(d_j+d_k)}. \tag{A.12}$$

□

# B Fourier series of trigonometric power-law coefficients

In the next result, we establish the asymptotic behavior of the Fourier series of the trigonometric power-law coefficients (1.10). The proof is based on the work of Wainger (1965) who obtained a similar result for double-sided trigonometric power-law coefficients (Theorem 10 in Wainger (1965), p. 53). For shortness sake, we shall abbreviate the work of Wainger (1965) by WA.

**Theorem B.1** *Let $0 < a < 1$ and $0 < b \le 1 - \frac{1}{2}a$. For $\epsilon > 0$, define*

$$f_{\epsilon,1}(x) = \sum_{n=0}^{\infty} n^{-b} \cos(2\pi n^a) e^{2\pi i n x - \epsilon n}, \qquad f_{\epsilon,2}(x) = \sum_{n=0}^{\infty} n^{-b} \sin(2\pi n^a) e^{2\pi i n x - \epsilon n}.$$

*Then, the limits $f_j(x) = \lim_{\epsilon \downarrow 0} f_{\epsilon,j}(x)$, $j = 1, 2$, exist in the pointwise sense for $x \neq 0$. Moreover, $f_j(x)$ are continuous for $x \neq 0$, $j = 1, 2$, and*

$$f_1(x) = |x|^{-d} e^{-\operatorname{sign}(x) i (\xi_a |x|^{-\frac{a}{1-a}} + \psi)} \left( c_{a,b} + O(|x|^{\frac{a}{1-a}}) \right) + C_1(x), \tag{B.1}$$

$$f_2(x) = |x|^{-d} \operatorname{sign}(x) i e^{-\operatorname{sign}(x) i (\xi_a |x|^{-\frac{a}{1-a}} + \psi)} \left( c_{a,b} + O(|x|^{\frac{a}{1-a}}) \right) + C_2(x), \tag{B.2}$$

*where $d = \frac{1-b-\frac{a}{2}}{1-a}$, $c_{a,b} = \frac{1}{2} a^{-\frac{b-1/2}{1-a}} (1-a)^{-1/2}$, $\xi_a = 2\pi(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}})$, $\psi = -\frac{\pi}{4}$ and $C_1(x)$, $C_2(x)$ are continuous functions.*

Proof: We follow to the extent possible the notation of Wainger (1965), abbreviated WA below. Consider the functions

$$\Phi_1(u) = \psi(u)|u|^{-b} \cos(2\pi |u|^a), \qquad \Phi_2(u) = \psi(u)|u|^{-b} \sin(2\pi |u|^a),$$

where $\psi(u) \in C^{\infty}(-\infty,\infty)$, $\psi(u) = 0$ for $u \le 1/2$, $\psi(u) = 1$ for $u \ge 1$ and $0 \le \psi(u) \le 1$. Let $\epsilon > 0$ and $D_{\epsilon,j}(x) = \{\Phi_j(u) e^{-\epsilon u}\}^{\vee}(x) = \int_{\mathbb{R}} \Phi_j(u) e^{2\pi i x u - \epsilon u} du$ be the inverse Fourier transforms of $e^{-\epsilon u}\Phi_j(u)$, $j=1,2$. Observe that the functions $f_{\epsilon,j}(x)=\sum_{n=0}^{\infty}\Phi_j(n)e^{2\pi inx-\epsilon n}$ are discrete counterparts of $D_{\epsilon,j}(x)$. The proof below will show that

$$D_j(x)=\lim_{\epsilon\downarrow 0}D_{\epsilon,j}(x),\quad j=1,2, \tag{B.3}$$

exist and are continuous at $x\neq 0$, that $f_j(x)$ and $D_j(x)$ are equal up to a continuous function, and that $D_j(x)$, $j=1,2$, have the asymptotic behavior of the first terms on the right-hand sides in (B.1)–(B.2). We will use Lemma 11 (p. 37) and Theorem 9 (p. 41) in WA.

Consider the function $F_\epsilon^{k,a,b}(x)$ appearing in 2.4 of WA, p. 44,

$$F_\epsilon^{k,a,b}(x)=2\pi|x|^{\frac{1}{2}(2-k)}\int_0^\infty \psi(u)u^{-b+\frac{1}{2}k}e^{2\pi iu^a-\epsilon u}J_{\frac{1}{2}(k-2)}(2\pi|x|u)du, \tag{B.4}$$

where $J_\mu(x)$ is a Bessel function of the first kind. (See, for example, Korenev (2002) for more information on Bessel functions.) The function (B.4) is denoted by $F_\epsilon(x)$ in WA. We added the superscripts $k$, $a$ and $b$ to avoid confusion regarding the values of these parameters. By Lemma B.1 below,

$$D_{\epsilon,1}(x)=\frac{1}{2}\mathrm{Re}(F_\epsilon^{1,a,b}(x))+\mathrm{sign}(x)i\frac{1}{2}|x|\mathrm{Re}(F_\epsilon^{3,a,b+1}(x)), \tag{B.5}$$

$$D_{\epsilon,2}(x)=\frac{1}{2}\mathrm{Im}(F_\epsilon^{1,a,b}(x))+\mathrm{sign}(x)i\frac{1}{2}|x|\mathrm{Im}(F_\epsilon^{3,a,b+1}(x)). \tag{B.6}$$

By Theorem 9 in WA (see again 2.4 in WA, p. 44), $F^{k,a,b}(x)=\lim_{\epsilon\downarrow 0}F_\epsilon^{k,a,b}(x)$ exists in the pointwise sense for $x\neq 0$, and $F^{k,a,b}(x)$ is continuous for $x\neq 0$. Thus, in view of (B.5) and (B.6), the same holds for $D_{\epsilon,j}(x)$ and $D_j(x)$, $j=1,2$.

We now want to use Lemma 11 in WA, p. 37, with $\Phi=\Phi_j$, $F_\epsilon=D_{\epsilon,j}$ and $F=D_j$ in the lemma. The established relation (B.3) is one of the assumptions of the lemma. The other assumptions are $|\Phi_j(u)|=O(e^{\epsilon|x|})$ as $u\to\infty$, and $|D_{\epsilon,j}(x)|=O(|x|^{-1-\mu})$ uniformly in $\epsilon$ for some $\mu>0$, as $x\to\infty$. The first of these assumptions certainly holds. The second assumption can be verified by using Theorem 9 in WA. Thus, Lemma 11 of WA yields that the limits $f_j(x)$ exist, are continuous at $x\neq 0$, and are equal to $D_j(x)$ up to continuous functions.

It remains to show that the functions $D_j(x)$ behave as the first terms on the right-hand sides of (B.1)–(B.2). Theorem 9, ii), in WA shows that, for $b\leq k(1-\frac{1}{2}a)$ and $m_0=0$,

$$F^{k,a,b}(x)=|x|^{-\frac{k-b-\frac{ka}{2}}{1-a}}e^{i\xi_a|x|^{-\frac{a}{1-a}}}\left(\alpha_0+O(|x|^{\frac{a}{1-a}})\right)+\widetilde{C}_k(|x|), \tag{B.7}$$

where $\widetilde{C}_k(|x|)$ is a continuous function and $\alpha_0\in\mathbb{C}\backslash\{0\}$ depends on $a$, $b$ and $k$. For the asymptotic behavior of (B.1)–(B.2), we need and exact form of the constant $\alpha_0$ when $k=1$ and $k=3$. By using a version of the saddle point method, which is finer than the one used in Lemma 13 of WA, we show in Lemma B.2 below that

$$\alpha_0=2c_{a,b}e^{i\psi} \tag{B.8}$$

when $k=1$, and

$$\alpha_0=2c_{a,b}ie^{i\psi} \tag{B.9}$$

when $k=3$. This yields

$$F^{1,a,b}(x)=|x|^{-d}e^{i(\xi_a|x|^{-\frac{a}{1-a}}+\psi)}\left(2c_{a,b}+O(|x|^{\frac{a}{1-a}})\right)+\widetilde{C}_1(|x|), \tag{B.10}$$

$$F^{3,a,b+1}(x)=|x|^{-d-1}ie^{i(\xi_a|x|^{-\frac{a}{1-a}}+\psi)}\left(2c_{a,b}+O(|x|^{\frac{a}{1-a}})\right)+\widetilde{C}_3(|x|). \tag{B.11}$$

Letting $\epsilon \downarrow 0$ in (B.5), (B.6), and using (B.10) and (B.11), we conclude that

$$D_1(x) = |x|^{-d} e^{-\operatorname{sign}(x) i(\xi_a |x|^{-\frac{a}{1-a}} + \psi)} \left( c_{a,b} + O(|x|^{\frac{a}{1-a}}) \right) + \frac{1}{2} \left( \operatorname{Re}(\widetilde{C}_1(|x|)) + i|x| \operatorname{Re}(\widetilde{C}_3(|x|)) \right),$$

$$D_2(x) = |x|^{-d} \operatorname{sign}(x) i e^{-\operatorname{sign}(x) i(\xi_a |x|^{-\frac{a}{1-a}} + \psi)} \left( c_{a,b} + O(|x|^{\frac{a}{1-a}}) \right) + \frac{1}{2} \left( \operatorname{Im}(\widetilde{C}_1(|x|)) + i|x| \operatorname{Im}(\widetilde{C}_3(|x|)) \right),$$

where we also used the identities

$$\cos(y) - \operatorname{sign}(x) i \sin(y) = e^{-\operatorname{sign}(x) iy}, \quad \sin(y) + \operatorname{sign}(x) i \cos(y) = \operatorname{sign}(x) i e^{-\operatorname{sign}(x) iy}.$$

This completes the proof. $\square$

The next two auxiliary lemmas were used in the proof of Theorem B.1 above.

**Lemma B.1** *The functions $D_{\epsilon,j}(x)$, $j = 1, 2$, $F_\epsilon^{k,a,b}(x)$, defined in (B.3) and (B.4), satisfy the relations (B.5) and (B.6).*

PROOF: By using the Bessel function properties $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$ and $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$ (Korenev (2002), p. 16) and the facts that $\cos(|x|) = \cos(x)$ and $\sin(|x|) = \operatorname{sign}(x) \sin(x)$, we have

$$F_\epsilon^{1,a,b}(x) = 2 \int_0^\infty \psi(u) u^{-b} e^{2\pi i u^a - \epsilon u} \cos(2\pi x u) du,$$

$$F_\epsilon^{3,a,b+1}(x) = 2\operatorname{sign}(x) |x|^{-1} \int_0^\infty \psi(u) u^{-b} e^{2\pi i u^a - \epsilon u} \sin(2\pi x u) du.$$

Since $\psi(u) = 0$ for $u < 0$, we can also rewrite the inverse Fourier transforms $D_{\epsilon,j}(x)$ as

$$D_{\epsilon,1}(x) = \int_0^\infty \psi(u) u^{-b} \cos(2\pi u^a) e^{2\pi i x u - \epsilon u} du, \qquad D_{\epsilon,2}(x) = \int_0^\infty \psi(u) u^{-b} \sin(2\pi u^a) e^{2\pi i x u - \epsilon u} du.$$

Since

$$\operatorname{Re}(D_{\epsilon,1}(x)) = \frac{1}{2} \operatorname{Re}(F_\epsilon^{1,a,b}(x)), \quad \operatorname{Im}(D_{\epsilon,1}(x)) = \frac{1}{2} \operatorname{sign}(x) |x| \operatorname{Re}(F_\epsilon^{3,a,b+1}(x)),$$

$$\operatorname{Re}(D_{\epsilon,2}(x)) = \frac{1}{2} \operatorname{Im}(F_\epsilon^{1,a,b}(x)), \quad \operatorname{Im}(D_{\epsilon,2}(x)) = \frac{1}{2} \operatorname{sign}(x) |x| \operatorname{Im}(F_\epsilon^{3,a,b+1}(x)),$$

we conclude that (B.5) and (B.6) hold. $\square$

The second auxiliary lemma uses a saddle point method. The saddle point method allows computing asymptotic expansions of integrals of the form

$$I(t) = \int_C f(z) e^{t h(z)} dz, \quad \text{as} \quad t \to \infty,$$

where $C$ is a contour in the complex plane and the functions $f(z)$, $h(z)$ are holomorphic in a neighborhood of this contour. According to the method, if $z_0$ is an interior point of $C$ and a saddle point of $h(z)$, that is, $h'(z_0) = 0$, $h''(z_0) \neq 0$, then

$$I(t) = \sqrt{\frac{2\pi}{-h''(z_0)}} t^{-1/2} e^{t h(z_0)} \Big( f(z_0) + O(t^{-1}) \Big), \quad \text{as} \quad t \to \infty. \tag{B.12}$$

See, for example, Fedoryuk (2011). The version of the saddle point result (B.12) used by WA, Lemma 13, pp. 42-43, provides only the absolute value of the constant at $t^{1/2} e^{th(z_0)}$ in (B.12), that is, the value $(2\pi)^{1/2} |h''(z_0)|^{-1/2} |f_{z_0}|$. (This is also after correcting the typo in WA, p. 43, where the exponent 1/2 of $|h''(\xi)|$ should be replaced by $-1/2$.) The finer version (B.12) allows us to identify the constant $\alpha_0$ in (B.7) as stated in the next lemma.

**Lemma B.2** *The coefficient $\alpha_0$ appearing in (B.7) is given by (B.8) and (B.9) when $k = 1$ and 3, respectively, and the relations (B.10) and (B.11) hold.*

Proof: To prove that $F^{k,a,b}(x)$ in (B.7) is the limit of $F_\epsilon^{k,a,b}(x)$, Wainger (1965) decomposes $F_\epsilon^{k,a,b}(x)$ into several integrals. The main contribution to $F^{k,a,b}(x)$ comes from the integral given in 2.20 on p. 49 in WA,

$$H_{II}^{k,a,b}(x) = 2|x|^{\frac{b-k}{1-a}} t^{\frac{1}{2}(1-k)} \int_{II} s^{-b+\frac{1}{2}(k-1)} e^{th_1(s)} S_1^{\frac{1}{2}(k-2)}(2\pi st)ds,$$

where $h_1(s) = 2\pi i s^a - 2\pi i s$, $t = |x|^{-\frac{a}{1-a}}$, $S_1^\mu(z)$ is an analytic function given in Lemma 12 of WA, and $II$ is the contour described on p. 47 of WA. Therefore, it is enough to show that $H_{II}^{1,a,b}(x)$ and $H_{II}^{3,a,b+1}(x)$ are equal to the right-hand sides of (B.10) and (B.11), respectively, up to a continuous function. When $k = 1$ and $k = 3$, we get from Lemma 12 of WA that $S_1^{-1/2}(z) = 1/2$, $S_1^{1/2}(z) = i/2$. Then,

$$H_{II}^{1,a,b}(x) = |x|^{\frac{b-1}{1-a}} \int_{II} s^{-b} e^{th_1(s)} ds, \qquad H_{II}^{3,a,b+1}(x) = |x|^{\frac{b-2+a}{1-a}} i \int_{II} s^{-b} e^{th_1(s)} ds.$$

(Note that the second term is with $b+1$ to correspond to $F^{3,a,b+1}(x)$ in (B.11)). Next, consider the integral $I(t) = \int_{II} s^{-b} e^{th_1(s)} ds$. Let $\xi = a^{\frac{1}{1-a}}$ and observe that the point $(\xi, 0)$ is a saddle point of $h_1$ that lies in the interior of the contour $II$ (as seen in figure 1 of WA, p. 46). Then, from (B.12), $I(t) = t^{-\frac{1}{2}} e^{i(t\xi_a + \psi)} \left(c_{a,b} + O(t^{-1})\right)$, as $t \to \infty$. This yields

$$H^{1,a,b}(x) = |x|^{-d} e^{i(\xi_a |x|^{-\frac{a}{1-a}} + \psi)} \left(2c_{a,b} + O(|x|^{\frac{a}{1-a}})\right),$$

$$H^{3,a,b+1}(x) = |x|^{-d-1} i e^{i(\xi_a |x|^{-\frac{a}{1-a}} + \psi)} \left(2c_{a,b} + O(|x|^{\frac{a}{1-a}})\right).$$

□

Finally, we include the following elementary lemma which is used in the proofs of Propositions 4.1 and 4.2.

**Lemma B.3** *Let $a, b$ and $f_1, f_2$ be as in Theorem B.1. If*

$$b > \frac{1}{2},$$

*then the trigonometric power-law coefficients (1.10) are in $l^2(\mathbb{Z})$, and their Fourier series (defined in the $L^2(-1,1]$-sense) satisfy*

$$f_1(x) = \sum_{n=0}^{\infty} \cos(2\pi n^a) n^{-b} e^{2\pi i n x}, \qquad f_2(x) = \sum_{n=0}^{\infty} \sin(2\pi n^a) n^{-b} e^{2\pi i n x}, \qquad a.e.\ dx. \tag{B.13}$$

Proof: Consider the functions $f_{\epsilon,j}(x)$, $j = 1, 2$, defined in Theorem B.1. Since

$$\int_{-1/2}^{1/2} \left| f_{\epsilon,1}(x) - \sum_{n=0}^{\infty} \cos(2\pi n^a) n^{-b} e^{2\pi i n x} \right|^2 dx = \sum_{n=0}^{\infty} n^{-2b} \cos^2(2\pi n^a)(e^{-\epsilon n} - 1)^2 \to 0,$$

as $\epsilon \downarrow 0$, we have $f_{\epsilon,1}(x)$ converging to the Fourier series $\sum_{n=0}^\infty \cos(2\pi n^a) n^{-b} e^{2\pi i n x}$ in the $L^2(-1/2, 1/2]-$sense. By Theorem B.1, $f_{\epsilon,1}(x)$ to $f_1(x)$ pointwise. The uniqueness of the limit yields the first relation in (B.13). The proof of the second relation in (B.13) is analogous. □

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