Multivariate count time series with flexible autocovariances

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Outline

Part 1

- Motivation for a bivariate count time series model
- Review of long memory

Part 2

- Model construction
- Quasi-maximum likelihood inference
- Data application
- Future work

Motivations from data

Examples of correlated counts observed at different time points

- Number of patients with different but related symptoms
- Occurrences of physical phenomena at different locations
- Number of occurrences of various crime types
- Number of trades of different stocks in a portfolio

Capturing both serial and cross-correlation complicates model specification and inference.

Motivations from data

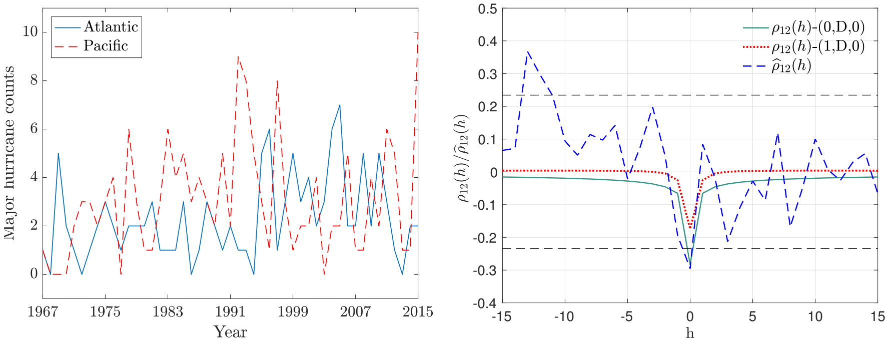


Figure: Left: Annual number of Saffir-Sampson category 3 and stronger hurricanes in North Pacific and North Atlantic Basins. Right: Theoretical/sample (red, green/blue) ccfs of the numbers of major hurricanes in the Atlantic and the Pacific.

- Negative cross correlation
- NA consistent with Poisson assumption, NP somewhat overdispersed

Motivation from data

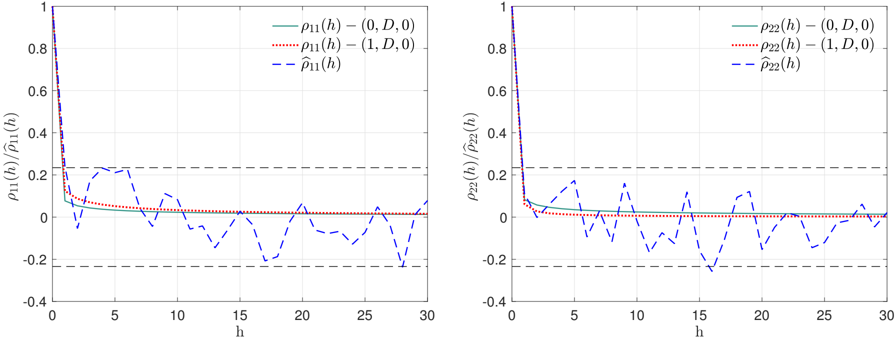


Figure: Sample (blue lines) and theoretical (red, green lines) auto-correlation functions of major hurricane counts in the Atlantic and Pacific Basins.

- Slow decay of dependence?

We would like to have a model that captures all 3 features.

Literature

Bivariate Poisson $BP(\theta_1, \theta_2, \theta_0)$

$$P(y_1, y_2) = e^{-(\theta_1+\theta_2+\theta_0)} \frac{\theta_1{}^{y_1}}{y_1!} \frac{\theta_2{}^{y_2}}{y_2!} \sum_{s=0}^{\min(y_1,y_2)} \binom{y_1}{s} \binom{y_2}{s} s! \left(\frac{\theta_0}{\theta_1 \theta_2}\right)^s$$

- Poisson$(\theta_i + \theta_0)$ marginals, covariance θ_0
- Computationally intensive for large counts, positive correlation
- $BP(\theta_1, \theta_2, 0)$, $(\theta_1, \theta_2) \sim$ logNormal. Positive and negative correlation but marginals require numerical integration
- multivariate case?

Literature

- INAR(1) model

$$Y_t = \alpha \circ Y_{t-1} + R_t,$$

$\alpha \in [0,1]$, R_t uncorrelated, nonnegative, integer-valued random variables. The thinning operator $\circ$ is defined as

$$\alpha \circ Y = \sum_{i=1}^{Y} Z_i, \quad Z_i \sim \text{Bernoulli}(\alpha)$$

- BINAR(1)

$$\begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \circ \begin{bmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{bmatrix} + \begin{bmatrix} R_{1t} \\ R_{2t} \end{bmatrix}$$

- positive mean, variance and autocovariance

Some notation

- $X = \{X_n\}_{n\in\mathbb{Z}}$ is a zero mean, second-order stationary time series
- $\rho(k) = \mathbb{E}X_nX_{n+k}, k \in \mathbb{Z}$, is the autocorrelation function of X
- $\{\epsilon_n\}_{n\in\mathbb{Z}}$ is a WN series with $\mathbb{E}\epsilon_n^2 = 1$.
- $a_n \sim b_n$ implies that $a_n/b_n \to 1$ as $n \to \infty$.

The definitions and models of the time series we will discuss involve the so-called *long-range dependent* parameter $d \in (0, 1/2)$.

Univariate long-range dependent series

Consider the following two **non-equivalent** conditions:

I. X_n admits the form $X_n = \sum_{k=0}^{\infty} \psi_k \epsilon_{n-k}, \quad \psi_k \underset{k\to\infty}{\sim} c_1 k^{d-1}$

II. The acf of X_n satisfies $\rho(k) \sim c_2 k^{2d-1}$ as $k \to \infty$.

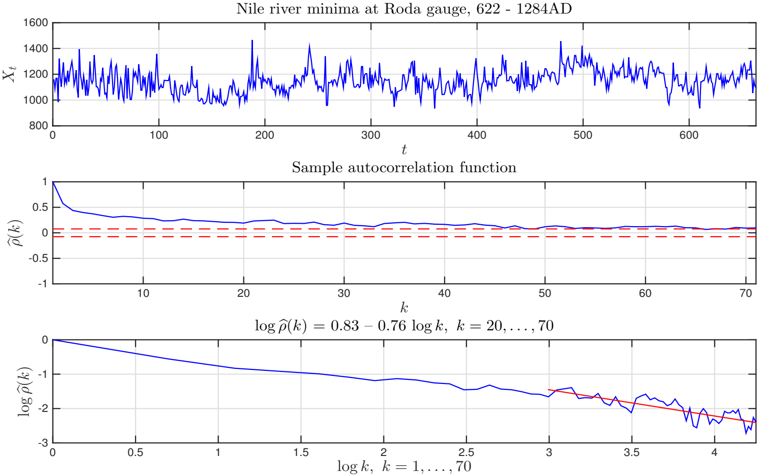
- X_n is called *long-range dependent* if one of the conditions I–II holds.
- Conditions I,II, imply that $\sum_{k=-\infty}^{\infty} |\rho(k)| = \infty$.
- A series with $\sum_{k=-\infty}^{\infty} |\rho(k)| < \infty$ is called *short-range dependent*.

LRD vs SRD

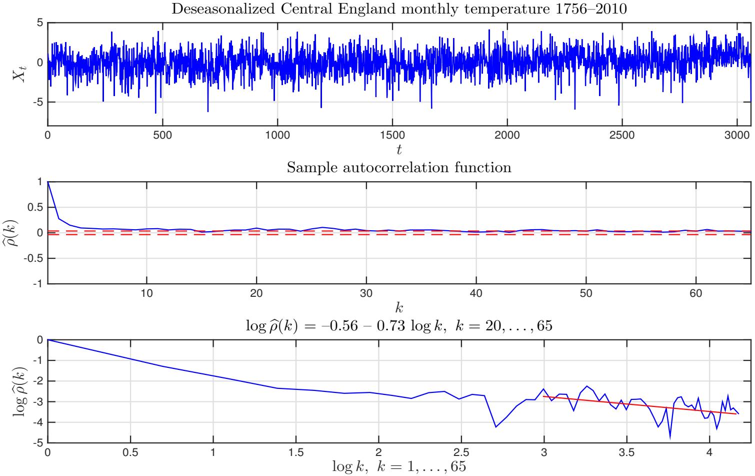
Characterization	LRD	SRD
$\sum_{k=-\infty}^{\infty} \|\rho(k)\|$	∞	$< \infty$
$\rho(k)$	k^{2d-1}	$O(e^{-\alpha k})$
$\sum_{k=0}^{\infty} \psi_k \epsilon_{n-k}$	$\psi_k \sim k^{d-1}$	$O(e^{-\alpha k})$
$\mathrm{Var}(\overline{X}_N)$	N^{2d-1}	N^{-1}

Question: How does the last relation affect inference?

Some examples of LRD series



Some examples of LRD series



ARFIMA$(0, d, 0)$ - A parametric LRD model

- If $d \in \mathbb{Z}^+$, then $\{X_n\}_{n\in\mathbb{Z}}$ is said to be an $ARIMA(0, d, 0)$ process if

$$(I - B)^d X_n = \epsilon_n, \tag{1}$$

- If $d \in (0, 1/2)$ we interpret the solution of (1) as

$$X_n = (I - B)^{-d}\epsilon_n = \sum_{k=0}^{\infty} b_k B^k \epsilon_n = \sum_{k=0}^{\infty} b_k \epsilon_{n-k},$$

- $b_k's$ are the Taylor coefficients of $(1 - z)^{-d}$ and satisfy

$$b_k = \frac{\Gamma(k + d)}{\Gamma(k + 1)\Gamma(d)} \underset{k\to\infty}{\sim} k^{d-1}, \quad \sum_{k=0}^{\infty} b_k^2 < \infty.$$

Bivariate long-range dependent series

- $\{X_n\}_{n\in\mathbb{Z}}=\{(X_{1,n},X_{2,n})'\}_{n\in\mathbb{Z}}$ is a bivariate, zero mean, second-order stationary time series
- $\rho_{ij}(k) = \mathbb{E}X_{i,n}X_{j,n+k},\ i,j = 1,2$ is the (i,j) component of the autocorrelation matrix function
- $\{\epsilon_n\}_{n\in\mathbb{Z}}=\{(\epsilon_{1,n},\epsilon_{2,n})'\}_{n\in\mathbb{Z}}$ is a bivariate WN with $\mathbb{E}\epsilon_n\epsilon_n' = \Sigma$.

The definitions and models of the time series we will discuss involve the so-called *long-range dependent* parameters $d_1, d_2 \in (0, 1/2)$.

Definitions of bivariate LRD series

- Time domain definition of bivariate LRD

$$\rho_{ij}(h) \sim R_{ij} h^{d_i+d_j-1}, \quad h \to \infty,$$

where $R_{ij} \in \mathbb{R}$, $i, j = 1, 2$.

- VARFIMA$(0, D, 0)$ model, $D = \text{diag}(d_1, d_2)$.

$$X_n = \begin{pmatrix} X_{1,n} \\ X_{2,n} \end{pmatrix} = \begin{pmatrix} (I-B)^{-d_1}\epsilon_{1,n} \\ (I-B)^{-d_2}\epsilon_{2,n} \end{pmatrix} = (I-B)^{-D}\epsilon_n,$$

- The acmf has the exact form

$$\rho_{ij}^X(h) = \sigma_{ij} \frac{(-1)^h \Gamma(1-d_i-d_j)}{\Gamma(1-d_i+h)\Gamma(1-d_j-h)}, \; i, j = 1, 2.$$

Bivariate Count LRD model

1. *Let X_t be a bivariate LRD series* with acmf $\rho^X(h), h \in \mathbb{Z}$ and

$$X_t \sim N_2\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right), \quad \rho = \rho^X(0).$$

2. *Place the components of X into categories*:

$$S_t = (S_{1,t}, S_{2,t})'_{t\in\mathbb{Z}} = \left(1_{\{X_{1,t}>0\}}, 1_{\{X_{2,t}>0\}}\right)'_{t\in\mathbb{Z}},$$

Fact: The series $\{S_t\}_{t\in\mathbb{Z}}$ is stationary with $\mathbb{E}S_t = (1/2, 1/2)'$ and

$$\rho^S_{ij}(h) = \frac{1}{2\pi}\text{asin}(\rho^X_{ij}(h)), \quad h \in \mathbb{Z}.$$

Remark 1: $P(X_{i,t} > 0, X_{j,t+h} > 0) = \frac{1}{4} + \text{asin}(\rho^X_{ij}(h))/2\pi$.

Remark 2: $\text{asin}(z) \underset{z\to 0}{\sim} z$ and so S_t is LRD if X_t is LRD.

Bivariate Count LRD model

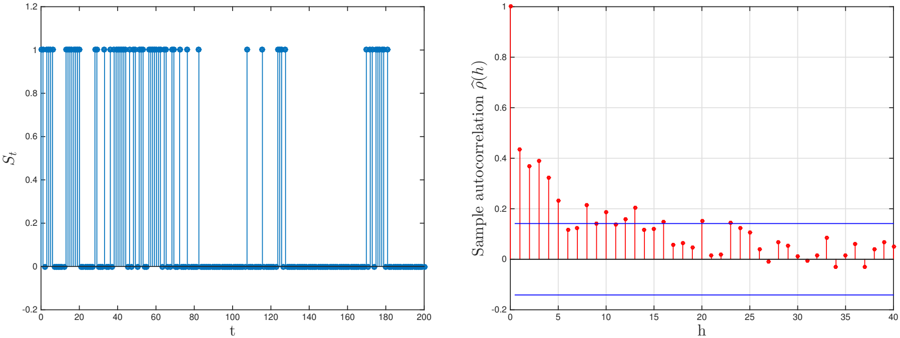


Figure: Left: One realization of the series $S_{1,t}$ with sample size $N = 200$. The underlying process $X_{1,t}$ is a Gaussian FARIMA$(0, 0.4, 0)$. Right: Sample acf $\widehat{\rho}^S(h)$ for lags $h = 0, \dots, 40$.The blue lines indicate the 95% confidence interval.

Binomial marginal distributions

3. *Superimpose IID copies of S_t*

   Binomial marginals

   Let $\{S_{1,t}^{(k)}, S_{2,t}^{(k)}\}_{k=1}^{\infty}$ be a sequence of IID copies of S_t. Then the series Y_t with components

$$Y_{i,t} = \sum_{k=1}^{M} S_{i,t}^{(k)}, \quad i = 1, 2,$$

   has *Binomial*$(M, 1/2)$ univariate marginal distributions with

$$\rho^Y(h) = M\rho^S(h), \quad h \in \mathbb{Z}.$$

   Long memory of S_t implies that Y_t is also LRD.

Poisson marginal distributions

Poisson marginals

Consider the series $\{Y_t\}_{t\in\mathbb{Z}} = \{(Y_{1,t}, Y_{2,t})'\}_{t\in\mathbb{Z}}$ with

$$Y_{i,t} = \sum_{k=1}^{N_{i,t}} S_{i,t}^{(k)}, \quad N_{i,t} \overset{\text{ind}}{\sim} \text{Poisson}(\lambda_i), \quad \lambda_i > 0.$$

Fact: The series Y_t is stationary with $\mathbb{E}Y_t = (\lambda_1/2, \lambda_2/2)'$ and

$$\rho_{ij}^{Y}(h) = \frac{1}{2\pi} c_{ij} \text{asin}(\rho_{ij}^{X}(h)),$$

$$c_{ij} = \begin{cases} 2\lambda_i, & i = j, h = 0, \\ \lambda_i F_{W_{ij}}(-1) + \lambda_j(1 - F_{W_{ij}}(1)), & i \neq j, \end{cases}$$

where $W_{ij} \sim \text{Skellam}(\lambda_i, \lambda_j) = \text{Poisson}(\lambda_i) - \text{Poisson}(\lambda_j)$.

Sketch Proof

Joint pmf of $N_{i,t}, N_{j,t+h}$: $p_{ij}(h) = P(N_{i,t} = n_i, N_{j,t+h} = n_j)$.

$$
\begin{aligned}
\mathbb{E} Y_{i,t} Y_{j,t+h} &= \mathbb{E}\Big(\mathbb{E} \sum_{m=1}^{n_i} S_{i,t}^{(m)} \sum_{k=1}^{n_j} S_{j,t+h}^{(k)} \Big| N_{i,t} = n_i, N_{j,t+h} = n_j\Big) \\
&= \sum_{n_i, n_j = 0}^{\infty} \mathbb{E}\Big(\sum_{m=1}^{n_i} S_{i,t}^{(m)} \sum_{k=1}^{n_j} S_{j,t+h}^{(k)} \Big| N_{i,t} = n_i, N_{j,t+h} = n_j\Big) p_{ij}(h) \\
&= \sum_{n_i, n_j = 0}^{\infty} \Big(M_{ij} \frac{\arcsin(\rho_{ij}^{X}(h))}{2\pi} + \frac{\Pi_{ij}}{4}\Big) p_{ij}(h) \\
&= \tfrac{1}{2\pi} \arcsin(\rho_{ij}^{X}(h)) \mathbb{E} \min(N_{i,t}, N_{j,t+h}) + \tfrac{1}{4} \mathbb{E} N_{i,t} N_{j,t+h}.
\end{aligned}
$$

- $M_{ij} = \min(n_i, n_j)$ cross products of $S_{i,t}^{(m)}$ and $S_{j,t+h}^{(k)}$ for $m = k$
- $\Pi_{ij} - M_{ij}$ cross products of $S_{i,t}^{(m)}$ and $S_{j,t+h}^{(k)}$ for $m \neq k$, $\Pi_{ij} = n_i n_j$

Statistical Inference

Let $\mathbf{Y}_N = (Y_1 \ldots, Y_N)'$ with $\Gamma_N = \mathbb{E}\mathbf{Y}_N\mathbf{Y}_N'$.

- Gaussian log-Likelihood

$$\ell(\Gamma_N) \propto -\frac{1}{2}\log|\Gamma_N| - \frac{1}{2}\mathbf{Y}^T\Gamma_N^{-1}\mathbf{Y}_N$$

- It can be rewritten as

$$\ell(\theta;\mathbf{Y}_N) \propto -\frac{1}{2}\sum_{j=1}^{N}\log|V_{j-1}| - \frac{1}{2}\sum_{j=1}^{N}(Y_j - \widehat{Y}_j)'V_{j-1}^{-1}(Y_j - \widehat{Y}_j),$$

where $\widehat{Y}_j = \mathbb{E}(Y_j|Y_1,\ldots,Y_{j-1})$, $V_{j-1} = \mathbb{E}(Y_j - \widehat{Y}_j)(Y_j - \widehat{Y}_j)'$

- $\widehat{Y}_j$, V_{j-1} can be computed can be computed from $\rho^Y(h)$ in $O(N^2)$.

Statistical Inference

Let $\Phi(z), \Theta(z)$ be the typical AR, MA polynomials with orders p, q

LRD model: $\Phi(B)X_n = (I - B)^{-D}\Theta(B)\epsilon_n$, with $p, q \leq 1$

Parameter: $\theta = (d_1, d_2, \lambda_1, \lambda_2, \rho, \text{vec}(\Phi), \text{vec}(\Theta))$.

Quasi-MLE: $\widehat{\theta} = \underset{\theta}{\operatorname{argmax}}\ \ell(\theta; Y)$

Computational issues

- The entries of Σ, Φ, Θ appear in nonlinear constraints.
- We assumed marginal unit variances and a prescribed correlation for X. What parameters σ_{ij} ensure that?
- What if the desired σ_{ij} do not satisfy the constraints?

Simulation-$X_t \sim$VARFIMA$(0, D, 0)$

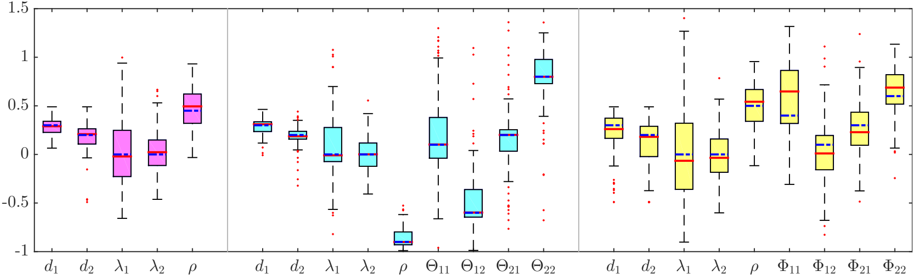


Figure: Boxplots of the estimates for $(p, q) = (0, 0)_1$ (left box), $(0, 1)$ (middle box) and $(1, 0)$ (right box) $T = 400$ and 100 replications. The dashed blue lines correspond to the true parameter values, while the solid red lines are the medians.

Simulation results

(p,q)	$(0,0)$		$(1,0)$		$(0,1)$	
T	200	400	200	400	200	400
d_1	-0.011 0.082	-0.010 0.059	-0.029 0.138	-0.039 0.107	0.059 0.057	0.069 0.038
d_2	-0.026 0.093	0.009 0.065	-0.019 0.183	-0.190 0.179	0.071 0.070	0.085 0.047
λ_1	-0.046 0.273	-0.021 0.229	-0.049 0.309	-0.065 0.338	-0.024 0.231	-0.013 0.199
λ_2	0.029 0.157	0.023 0.125	-0.077 0.228	-0.035 0.165	-0.023 0.156	0.010 0.149
ρ	0.072 0.218	0.045 0.155	-0.002 0.196	0.042 0.163	0.060 0.110	-0.011 0.050
$\Phi_{1,1}/\Theta_{1,1}$			0.069 0.243	0.248 0.267	-0.004 0.021	-0.017 0.023
$\Phi_{1,2}/\Theta_{1,2}$			0.017 0.241	-0.090 0.178	0.165 0.085	0.145 0.086
$\Phi_{2,1}/\Theta_{2,1}$			-0.023 0.158	-0.071 0.168	0.130 0.069	0.133 0.047
$\Phi_{2,2}/\Theta_{2,2}$			0.052 0.148	0.088 0.140	0.029 0.020	0.032 0.022

Table: MB and MAD of estimated parameters. True parameter values are $d_1 = 0.3, d_2 = 0.2, \lambda_1 = 3, \lambda_2 = 2\ \rho = -0.9,\ \Phi_{1,1} = 0.4, \Phi_{1,2} = 0.1,\ \Phi_{2,1} = 0.3,$ $\Phi_{2,2} = 0.6,\ \Theta_{1,1} = 0.1, \Theta_{1,2} = -0.6,\ \Theta_{2,1} = 0.2, \Theta_{2,2} = 0.8.$

Data application

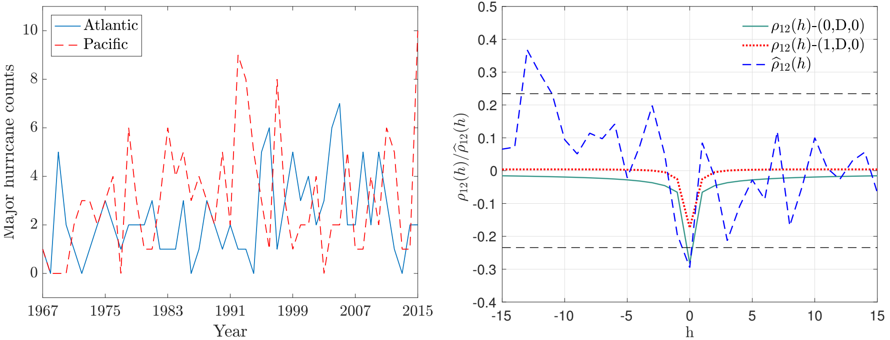


Figure: Left: Annual number of Saffir-Sampson category 3 and stronger hurricanes in NP and NA Basins. Right: Theoretical/sample ccfs

- $\bar{x}_{\text{Atl}} = 2.31, s^2_{\text{Atlantic}} = 2.97,\ \bar{x}_{\text{Pac}} = 3.10, s^2_{\text{Pacific}} = 5.76$
- $\text{IC}_{(0,D,0)} < \text{IC}_{(1,D,0)}$. For $(1, d, 0)$ $|\Sigma| < 0$ when $\rho < -0.7$.
- $\widehat{d}_1 = 0.24, \widehat{d}_2 = 0.23, \widehat{\lambda}_1 = 5.7, \widehat{\lambda}_2 = 11.5, \widehat{\rho} = -0.96$, $(\widehat{\rho}_Y = -0.28)$

Other marginal distributions

1. If $N_{i,t}$ ~Geo(α) then $Y_{i,t}$ ~Geo(γ), $\alpha = 2\gamma/(\gamma + 1)$.

2. Using the fact above we can also obtain a *NB* marginal.

3. For DU($\{u_1, \ldots, u_D\}$) pick $r_1, \ldots, r_D$

$$P(r_{k-1,i} < X_{i,t} < r_{k,i}) = \frac{1}{D}, \tag{2}$$

and take the series

$$U_{t,i} = \sum_{k=0}^{D-1} u_k 1_{[r_{k,i} < X_{i,t}. < r_{k+1,i}]}$$

For the latter we cannot use the quadrant Normal probabilities to obtain an exact form for the acv function.

Conclusions - Future work

1. Negative correlation, arbitrary marginals, LRD behavior
2. From bivariate to multivariate?
3. Other estimation methods (Pseudo-Likelihood, Bayesian, etc.)
4. Performance under misspecification and forecasting power

References

- Beran, et al., *Long-Memory Processes*. Springer, Heidelberg, 2013.
- Davis et al., eds. *Handbook of Discrete-Valued Time Series*. CRC Press, 2016.
- Livsey et al., *Multivariate count time series with flexible autocovariances*. Submitted