Latent Gaussian Count Time Series Modeling

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Outline

1. Motivation
2. Definitions and models
3. Statistical inference
4. Simulation performance and data application

Motivation - Dependent count time series

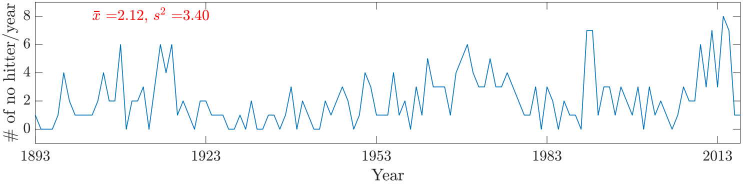
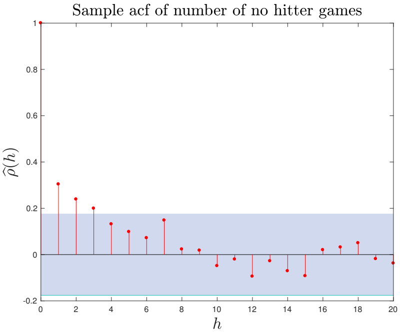
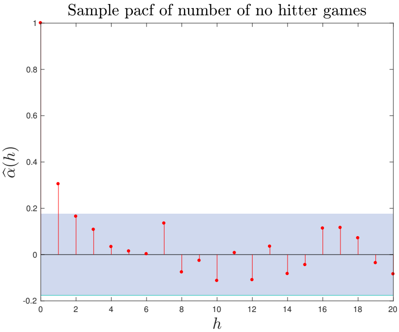


Figure: Yearly counts of no-hitter baseball games from 1875 to 2017.

- Over-dispersed time series of counts
- Negative Binomial, Generalized Poisson or some other distribution?

Motivation - Dependent count time series





- Possible dependence across time

We need count time series (cts) models that allow for flexible dependence structure and can produce any prescribed marginal distribution.

Copula transformation and latent Gaussian variable

- $\{Z_t\}_{t\in\mathbb{Z}}$: stationary, correlated, standard Gaussian series with cdf Φ.
  $\{X_t\}_{t\in\mathbb{Z}}$: stationary cts with desired marginal cdf $F(x) = \mathbb{P}(X_t \leq x)$.

- We model $\{X_t\}$ as

$$X_t = G(Z_t), \quad G(x) = F^{-1}(\Phi(x)), \quad x \in \mathbb{R}, \tag{1}$$

  where F^{-1} is the generalized inverse (quantile function) of F.

- By construction $\{X_t\}$ has marginal cdf F for each t.

How can we associate the dependences structure of $\{Z_t\}$ and $\{X_t\}$?

Linking dependence through acfs

- **Idea:** Try to link the acfs ρ_Z and ρ_X of the two series as

$$\rho_X(h) = \ell(\rho_Z(h))$$

through some function $\ell : [-1, 1] \to [-a, 1]$, $0 < a < 1$.

- ℓ should be $1 - 1$, feasibly computed, and yield large values of a.

- **Solution:** Expand G using *Hermite* polynomials (HP)

$$G(z) = \sum_{k=1}^{\infty} g_k H_k(z),$$

$$H_0 = 1, \quad H_1 = z, \quad H_2 = z^2 - 1, \quad H_3 = z^3 - 3z, \quad H_4 = z^4 - 6z^2 + 3.$$

HP properties

$$H_k(z) = (-1)^k e^{z^2/2} \frac{d^k}{dz^k} e^{-z^2/2},$$

1. HP form an orthogonal basis in $L^2(\mathbb{R}, \phi)$, $G(Z_t) = \sum_{k=1}^{\infty} g_k H_k(Z_t)$
2. $\text{Cov}(H_k(Z_t), H_k(Z_{t+h})) = k! \rho_Z(h)^k$
3. $\text{Cov}(G(Z_t), G(Z_{t+h})) = \sum_{k=0}^{\infty} k! g_k^2 \rho_Z(h)^k$, $\quad g_k = \frac{1}{k!} \mathbb{E}[G(Z_0) H_k(Z_0)]$

Link function & link coefficients

- We associate the acfs ρ_Z and ρ_X of the series $\{Z_t\}$ and $\{X_t\}$ as

$$\rho_X(h) = \sum_{k=1}^{\infty} \frac{k! g_k^2}{\sigma_X^2} \rho_Z(h)^k = \ell(\rho_Z(h)), \quad \ell(u) = \sum_{k=1}^{\infty} \ell_k u^k,$$

where $\ell(\cdot)$ and ℓ_k are called *link* function and *link* coefficients LC.

How flexible is the resulting dependence structure?

Link function properties

- Short memory in Z_t passes on to X_t
- Long memory in Z_t is also inherited in X_t for most marginals.
- $\ell(\cdot)$ yields the largest negative attainable correlation between two variables X_{t_1}, X_{t_2} with the same marginal distribution.

How should we calculate ℓ_k and ℓ?

Calculating LC

Letting $C_n = \mathbb{P}(X_t \leq n)$ and using HP properties we derive

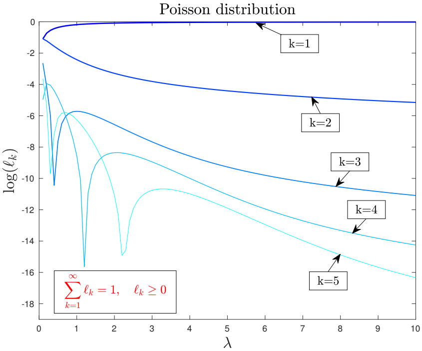
$$g_k = \frac{1}{k!\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{-\Phi^{-1}(C_n)^2/2} H_{k-1}(\Phi^{-1}(C_n)). \quad (2)$$

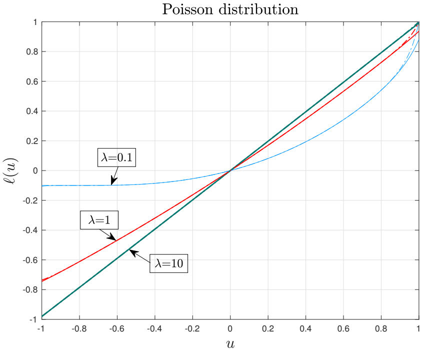
1. (2) converges for processes with finite variance (not obvious).
2. For fairly light-tailed C_n, $C_n \approx 1$ for $n > M$ for small/moderate M.
3. Truncation, HP asymptotics and Stirling's formula yield as $k \to \infty$

$$g_k(k!)^{1/2} \sim \frac{2^{-1/4}}{(\pi k)^{3/4}} \sum_{n=0}^{M-1} e^{-\frac{\Phi^{-1}(C_n)^2}{4}} \cos\left(\Phi^{-1}(C_n)\sqrt{k-1} - \frac{(k-1)\pi}{2}\right),$$

an approximation we found to be accurate even for moderate k.

Plotting LC and LF for Poisson(λ)





1. $\lambda \geq 1$: ℓ_1 carries all the *weight* of the LC (left) and $\ell(u) \approx u$ (right).
2. $\lambda < 1$: *weight* spreads across many LC negative ρ_X are impossible.

Inference

- $\boldsymbol{\theta}$, $\boldsymbol{\eta}$: marginal distribution and latent Gaussian acf parameters

- We approximate the likelihood

$$\mathcal{L}_T(\boldsymbol{\theta}, \boldsymbol{\eta}) = P(X_0 = x_0, X_1 = x_1, \ldots, X_T = x_T)$$

in two ways, through Gaussian likelihood and particle filtering (PF).

- To use PF approximation we write

$$\mathcal{L}_T(\boldsymbol{\theta}, \boldsymbol{\eta}) = \mathbb{P}(X_0 = x_0) \prod_{s=1}^{T} \mathbb{E}_X(w_s(\widehat{Z}_{s|s-1})), \tag{3}$$

where w_s is easily computed from DL quantities and the cdf of X_t.

Connection with HMM

1. When Z_t is an AR(p) process, our model is an HMM.

2. Generate particles Z_t^i and compute weights w_t^i, $i = 1, \dots, N$, using PF sampling algorithms (SIS, SISR, APF).

3. We can then approximate $\mathbb{E}_X[f(\widehat{Z}_{t|t+1})]$ for some function f, and $\mathcal{L}$ as

$$\widehat{\mathbb{E}}_X f(\widehat{Z}_{t+1|t}) = \frac{\frac{1}{N}\sum_{i=1}^N w_t^i f(\widehat{Z}_{t+1}^i)}{\frac{1}{N}\sum_{i=1}^N w_t^i},$$

$$\widehat{\mathcal{L}}_T(x_0, \dots, x_T) = \mathbb{P}(X_0 = x_0) \prod_{s=1}^T \widehat{\mathbb{E}}_X(w_s(\widehat{Z}_{s|s-1})),$$

Simulations-Poisson-AR(1)

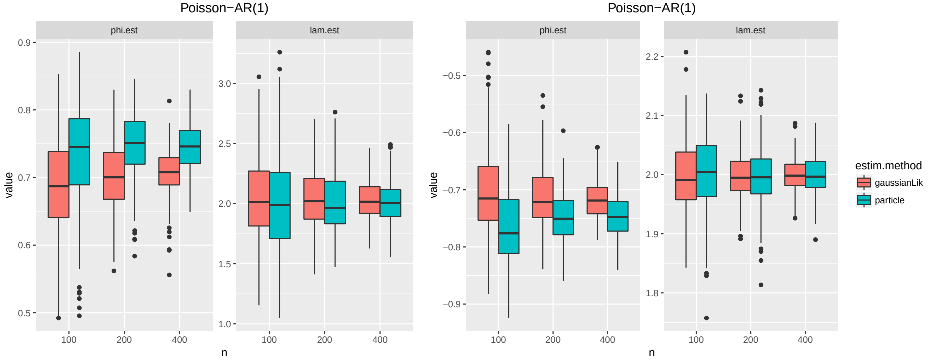
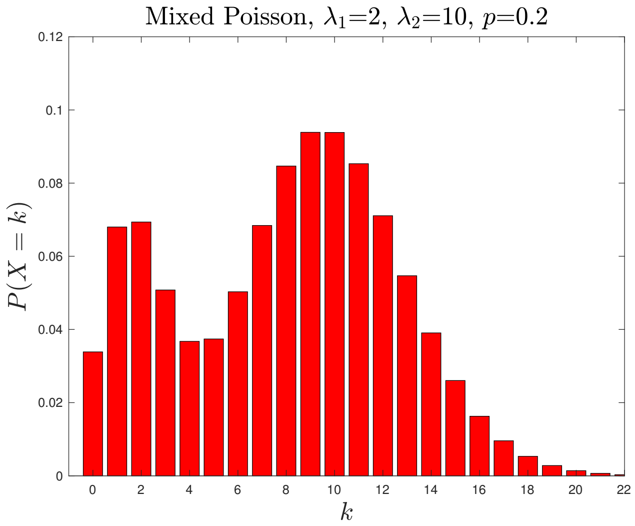


Figure: Estimates from simulated Poisson(2)–AR(1) series with true $\phi = 0.75$ (left) a and $\phi = -0.75$ (right) for sample sizes N= 100, 200, and 400.

Simulations-Mixed Poisson-AR(1)



Simulations-Mixed Poisson-AR(1)

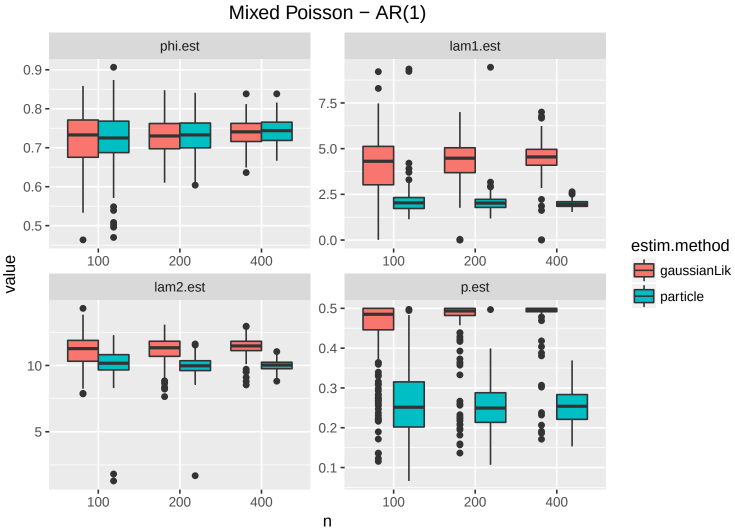


Figure: Estimates from Mixed Poisson(2,10)–AR(1) series with $p = 0.25, \phi = 0.75$

Application to no-hitter data

1. We use Gen. Poisson(η, λ) (overdispersion) and AR(1) (see pacf).
2. We also add two covariates: M_1 the # of games played in a season, and M_2 the height of the pitching mound to the model through

$$\lambda_t = \exp\left(\beta_0 + \beta_1 M_{1,t} + \beta_2 M_{2,t}\right)$$

Parameters	ϕ	β_0	β_1	β_2	η
GL Estimates	0.2665	-1.1496	0.7583	0.0338	0.1679
GL Standard Errors	0.0658	0.9069	0.2173	0.0436	0.0480

Parameters	ϕ	β_1	η
GL Estimates	0.2456	0.4059	0.1212
GL Standard Errors	0.0621	0.0480	0.0416

Summary

Cts model with a latent Gaussian variable

Flexible marginals and dependence structure

Connection with HMM and feasible inference

PIT and residual diagnostics

References

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Thank you!